

Multinomial forseg

Hvis vi har k mulige udfald af et forseg og et eksperiment består af n identiske uafh. forseg:

$$(x_1, \dots, x_k) \sim m(n, (\pi_1, \dots, \pi_k))$$

x_i antal i'te udfald, π_i ss for i'te udfald

tæthedsfkt. for $m(n, (\pi_1, \dots, \pi_k))$

$$\binom{n}{x_1, \dots, x_k} \pi_1^{x_1} \dots \pi_k^{x_k}, \quad x_i \leq n, \quad \pi_i \geq 0$$

I den fulde model er der $k-1$ frie parametre

i Hardy-Weinberg $((p^2, 2p(1-p), (1-p)^2))$ 1 fri parameter
og i den fuldt specificerende model 0 frie parametre

Vi ser at likelihood ratio testen for $M_{i-1} \rightarrow M_i$ ($e \geq 5$)

$$-2 \ln Q_i(x) = -2 \ln \frac{L_i(\pi(\theta))}{L_{i-1}(\pi(\psi))}$$

$$= 2 (l_{i-1}(\pi(\psi)) - l_i(\pi(\theta)))$$

$$= \sum_{j=1}^k x_j \ln \left(\frac{\pi_j(\psi)}{\pi_j(\theta)} \right) = \sum_{j=1}^k x_j \ln \left(\frac{e_j(\psi)}{e_j(\theta)} \right) \sim \chi^2(f_{i-1} - f_i)$$

I den fulde model: $e_j = x_j$ ($\pi_1^{x_1} \dots \pi_k^{x_k} \max_i (\frac{x_1}{x_{i-1}}, \frac{x_k}{x_i})$)

$$\text{Vi ser at } -2 \ln Q_i = -2 \ln Q_{0 \rightarrow i} - (-2 \ln Q_{0 \rightarrow i-1})$$

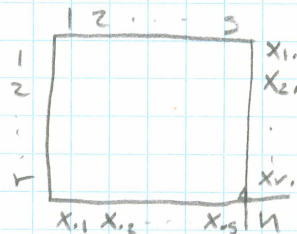
$$\Rightarrow -2 \ln Q_{0 \rightarrow i} = \sum_{j=1}^i -2 \ln Q_j$$

Hvis X_i iid $X_i \sim b(1, p) : \frac{\sum_{i=1}^n X_i - np}{\sqrt{np(1-p)}} \Rightarrow N(0,1)$

Dus. $b(n, p) \approx N(np, np(1-p))$

Dette kan benyttes til at angive konf. int.

Uafh. af inddelingskategorier:



$$\binom{n}{x_{11}, \dots, x_{rs}} \prod_{i,j} (p_i \sigma_j)^{x_{ij}}$$

$$(X_{1.}, \dots, X_{r.}) \sim m(n, (p_1, \dots, p_r))$$

$$\perp (X_{.1}, \dots, X_{.s}) \sim m(n, (\sigma_1, \dots, \sigma_s)) \quad \left(\begin{array}{l} \text{ihk} \perp \text{ i poisson} \\ \text{nu hvis } 1 \text{ r.} \end{array} \right)$$

H_1 : Full model $\rightarrow$ uafh. af inddelingskategorier.

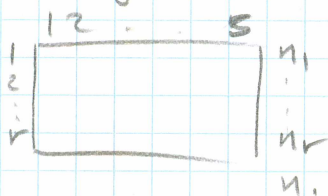
$$H_r: (p_1, \dots, p_r) = (p_1^0, \dots, p_r^0)$$

$$H_s: (\sigma_1, \dots, \sigma_s) = (\sigma_1^0, \dots, \sigma_s^0)$$

$$H_2: \pi_{ij} = p_i^0 \sigma_j^0$$

$$-2 \ln Q_c = -2 \ln Q_1 - 2 \ln Q_r - 2 \ln Q_s$$

Homogenitet



$\rightarrow$ Fisher's exact test
hypergeometrisk fordeling

binomial — poisson — neg. binomialfordeling

$$\text{Goodness of fit } (F_{\theta}(n_i) - F_{\theta}(n_{i-1}))$$

Poisson modeller

Lad $A \subseteq S$

Antag antal handels i A afh. kun af $|A|$

$A \cap B = \emptyset \Rightarrow$ antal handels i A uafh. antal handels i B

$$P(\text{antal handels i } A \geq 2) / |A| \rightarrow 0, |A| \rightarrow 0$$

Da $\exists \lambda > 0$ Antal handels i $A \sim \text{po}(\lambda |A|)$

$A_1, \dots, A_n$ partition af S : $(\frac{|A_1|}{|S|}, \dots, \frac{|A_n|}{|S|}) \sim m(\# \text{ handels i } S, \frac{|A_1|}{|S|}, \dots, \frac{|A_n|}{|S|})$

Hvis $X_i \sim \text{po}(\lambda)$ $i=1, \dots, n$

$$\frac{X. - n\lambda}{\sqrt{n\lambda}} \sim \approx N(0,1) \quad n \text{ stor}$$

Dvs. $X. \sim \approx N(n\lambda, n\lambda)$ hvilket kan benyttes til at konstruere et konf. int. for $n\lambda$ (derved λ) (betragt $\pm u_{0.975}$)

$X_{ij} \sim \text{po}(\lambda_i)$ $i=1, \dots, k$ $j=1, \dots, m_k$

$$L(\lambda_1, \dots, \lambda_n) = \prod_{j=1}^k e^{-\lambda_j} \lambda_j^{x_j} / x_j!$$

Likelihood ratio testoren $M_{i-1} \rightarrow M_i$

$$-2 \ln Q_{i-1,i} = -2 \ln \frac{L(\lambda(\hat{\theta}_i))}{L(\lambda(\hat{\theta}_{i-1}))} = 2(\ell_{i-1}(\lambda(\hat{\theta}_{i-1})) - \ell_i(\lambda(\hat{\theta}_i)))$$

$$= 2\left(\sum_{j=1}^k \lambda_j(\hat{\theta}_i) - \lambda_j(\hat{\theta}_{i-1}) + \sum_{j=1}^k x_j \ln(\lambda(\hat{\theta}_{i-1}) / \lambda(\hat{\theta}_i))\right)$$

$$= 2\left(\sum_{j=1}^k x_j \ln(e_{j,i-1} / e_{j,i}) + (e_{j,i} - e_{j,i-1})\right) \sim \chi^2(f_{i-1} - f_i)$$

$$e_{j,i} \geq 5$$

Idet $e^{-c\lambda} \lambda^x$ har max i $\lambda = x/c$ er

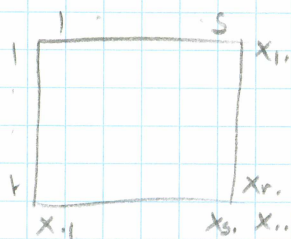
$$-2 \ln Q_{0,i} = 2 \left(\sum_{j=1}^k x_j \ln(x_j / e_{j,i}) + (e_{j,i} - x_j) \right)$$

Vi ser at $-2 \ln Q_{i,i} = -2 \ln Q_{0,i} - (-2 \ln Q_{0,i-1})$

$$-2 \ln Q_{q,i} = \sum_{j=1}^i -2 \ln Q_{j-1,j}$$

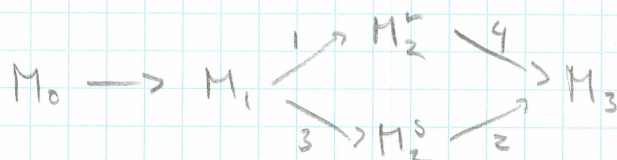
↑
asymptotisk uafh.

Multiplikative poissonmodel



$(X_{1.}, \dots, X_{r.})$ er ikke uafh.
af $(X_{1.}, \dots, X_{s.})$ (begge sum $X_{..}$)
betinges med $X_{..}$ f. uafh. og
multinomial modellen for uafh af inddeling.

$X_{ij} \sim \text{po}(\lambda p_i \pi_j)$ ($r+s-1$ frie parametre) $\hat{\lambda} = x_{..}$
 $\hat{p}_i = \frac{x_{i.}}{x_{..}}$
 $\hat{\pi}_j = \frac{x_{.j}}{x_{..}}$



1 det samme som 2

3 — 1 — 4

Proportional parametre $\rightarrow$ teste f.eks. homogenitet

Goodness of fit $(F_{\hat{\theta}}(r_i) - F_{\hat{\theta}}(r_{i-1}))$ el.

Fishers dispersionsindex $X_i \sim \text{po}(\lambda)$ $i=1, \dots, n$

$$\frac{s^2}{\bar{x}} \approx \chi^2(n-1)/(n-1)$$

$$\chi^2_{0.1025}(n-1)/(n-1), \chi^2_{0.975}(n-1)/(n-1)$$

$n \geq 15$ el. $\bar{x} \geq 5$

$$(s^2 = \frac{1}{n-1} (USS - \frac{s^2}{n}))$$

binomial — poisson — neg. binomial.

k normalfordelte observationsrækker, Bartlett's test

$$H_0: x_{ij} \sim N(\mu_i, \sigma_i^2) \quad , i=1, \dots, k \quad j=1, \dots, n_i$$

$$L(\mu, \sigma^2; x) = \prod_{i=1}^k \left(\frac{1}{2\pi\sigma_i^2} \right)^{n_i/2} e^{-\frac{1}{2\sigma_i^2} SSD_i}$$

$$y_i = \frac{SSD_{(i)}}{f_{(i)}} \sim \sigma_i^2 \chi^2(f_{(i)}) / f_{(i)} = \Gamma\left(\frac{f_{(i)}}{2}, \frac{f_{(i)}}{2\sigma_i^2}\right) \text{ uafh.}$$

$$L(\sigma^2) = \prod_{i=1}^k \Gamma\left(\frac{f_{(i)}}{2}\right)^{-1} \left(\frac{f_{(i)}}{2\sigma_i^2}\right)^{\frac{f_{(i)}}{2}} y_i^{\frac{f_{(i)}}{2}-1} e^{-\frac{f_{(i)}}{2\sigma_i^2} y_i}$$

$$\lambda \mapsto \lambda^x e^{-c\lambda} \text{ har max. i } \lambda = \frac{x}{c}, \text{ dvs.}$$

$$\left(\frac{f_{(i)}}{2\sigma_i^2}\right) = \frac{f_{(i)}/2}{y_i} \implies \hat{\sigma}_i^2 = y_i = \frac{SSD_{(i)}}{f_{(i)}} = s_{(i)}^2$$

$$\text{Under } H_0: \sigma_1^2 = \dots = \sigma_k^2 \text{ ses at } \hat{\sigma}^2 = \frac{\sum_{i=1}^k f_{(i)} SSD_{(i)}}{\sum_{i=1}^k f_{(i)}} = s_1^2$$

$$-2 \ln Q(y) = -2 \ln \frac{L(\hat{\sigma}^2)}{L(\sigma^2)}$$

$$= -2 \ln \left(\prod_{i=1}^k \left(\frac{\hat{\sigma}_i^2}{\sigma^2}\right)^{\frac{f_{(i)}}{2}} \right) = f_1 \ln s_1^2 - \sum_{i=1}^k f_{(i)} \ln s_{(i)}^2$$

Noget lignende ville fås hvis vi brugte den 'alm.' likelihood funkt.

Med korrektionen $C = 1 + \frac{1}{3(k-1)} \left(\sum_{i=1}^k \frac{1}{f_{(i)}} - \frac{1}{f_1} \right)$ bliver $\chi^2(k-1)$ app. til $-2 \ln Q(y)$ bedre.

I tilfældet hvor $X_j \sim N_{n_j \times p}(\mu_j, I_{n_j} \otimes \Sigma_j)$ $j=1, \dots, q$ er $-2 \ln Q(y)$ testst. for $\Sigma_1 = \dots = \Sigma_q$

fås noget 'tilsvarende' med udgangspunkt i de uafh.

$W(\Sigma_j, f_j)$ fordelte $SSD_{(i)} \sim$

Hvis hypotesen om ens varians accepteres

har vi en lin. normal model

$$\underline{x} \sim N\left(\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix}, I_n \otimes \sigma^2\right) \quad \mu_i = \begin{pmatrix} \mu_{i1} \\ \vdots \\ \mu_{in_i} \end{pmatrix}$$

$$H_0: \mu_1 = \dots = \mu_k$$

$$\hat{\mu} = \bar{x}_{..} \quad s^2 = \frac{SSD_{\text{om}}}{n.-1}, \quad SSD_{\text{om}} = SSD_1 + SSD_2 = \sum_{i,j} (x_{ij} - \bar{x}_{..})^2$$

$$SSD_2 = \sum_{i=1}^k n_i (\bar{x}_{i.} - \bar{x}_{..})^2$$

$$Q(x) = \left(\frac{SSD_1}{SSD_{\text{om}}} \right)^{n./2} = \left(\frac{1}{1 + \frac{SSD_2}{SSD_1}} \right)^{n./2}$$

$$s_2^2 = \frac{1}{k-1} SSD_2, \quad s_1^2 = \frac{1}{n.-k} SSD_1$$

Små værdier af $Q(x)$ er kritiske dvs. store værdier af $F = \frac{s_2^2}{s_1^2} \sim F(k-1, n.-k)$

$$E(x) = 1 - F_{F(k-1, n.-k)}(F)$$

Hvis hypotesen forkastes kan vi se på kontrasterne ($k(k-1)/2$ stk). $x_{i.} - x_{j.} \sim N(\mu_i - \mu_j, \sigma^2(\frac{1}{n_i} + \frac{1}{n_j}))$

Under hypotesen $\mu_i = \mu_j$ er $\frac{x_{i.} - x_{j.}}{\sqrt{s_1^2(\frac{1}{n_i} + \frac{1}{n_j})}} \sim t(n.-k)$

Diskussion af α , type I fejl og styrke

Styrken $1-\beta$ (β ss. type II fejl) vokser når ikke-centralitetsparameter

$$\frac{\mu_i - \mu_j}{\sigma \sqrt{1/n_i + 1/n_j}} \quad \text{vokser} \quad (\text{f.eks. er } \frac{1}{n_i} + \frac{1}{n_j} \text{ mindst når } n_i = n_j)$$

($\star$ er t fordelt med ikke-centralitetsparameter når $\mu_i \neq \mu_j$)

Statistisk inferens i lineære normale modeller

$$\underline{X} \sim N(\underline{\mu}, \sigma^2 \underline{I}_n)$$

$M_1: \underline{\mu} \in L_1$ lin. undrum af $\mathbb{R}^n$, $\dim L_1 = d_1$

$M_2: \underline{\mu} \in L_2$ —————, $\dim L_2 = d_2$

P_1 ortogonale projection på L_1

P_2 ————— L_2

$$SSD_1 := \|\underline{X} - P_1 \underline{X}\|^2, \quad SSD_2 = \|\underline{X} - P_2 \underline{X}\|^2, \quad SSD_{12} = \|P_1 \underline{X} - P_2 \underline{X}\|^2$$

Påstand:

Under M_1 er $\hat{\underline{\mu}} = P_1 \underline{X}$, $\hat{\sigma}^2 = SSD_1 / n$

$$S_1^2 = SSD_1 / (n - d_1) \sim \chi^2(n - d_1) / (n - d_1)$$

Under M_2 er $Q_{1 \rightarrow 2}$, $\hat{\underline{\mu}}$, S_1^2 uafh. og

$$\frac{SSD_{12} / (d_1 - d_2)}{SSD_1 / (n - d_1)} \sim F(d_1 - d_2, n - d_1)$$

Beweis:

Vælg en orthonormal basis $\underline{e}_1, \dots, \underline{e}_n$ for $\mathbb{R}^n$, så

$$\text{sp}(\underline{e}_1, \dots, \underline{e}_{d_1}) = L_1$$

$$\text{Sæt } A = \begin{pmatrix} -\underline{e}_1 - \\ -\underline{e}_n - \end{pmatrix}$$

$A\underline{X} \sim N(A\underline{\mu}, \sigma^2 \underline{I}_n)$, dvs. under L_1 er

$\underline{e}_i^* \underline{X} \sim N(0, \sigma^2)$, $i = d_1 + 1, \dots, n$ og

$$\|\underline{X} - P_1 \underline{X}\|^2 = \sum_{i=d_1+1}^n (\underline{e}_i^* \underline{X})^2 \sim \sigma^2 \chi^2(n - d_1)$$

$$\|P_1 \bar{X} - P_2 \bar{X}\|^2 = \sum_{i=d_2+1}^{d_1} (\underline{e}_i^T \bar{X})^2 \underset{\text{under } M_2}{\sim} \sigma^2 \chi^2(d_1 - d_2)$$

$$L(\mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2} \|\bar{X} - \mu\|^2}$$

Hvorf ser, at $\hat{\mu} = P_1 \bar{X}$ under M_1 og $\hat{\sigma}^2 = \frac{1}{n} SSD_1$ under M_1

$$\begin{aligned} Q_{1 \rightarrow 2} &= \left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} \right)^{\frac{n}{2}} = \left(\frac{\|\bar{X} - P_1 \bar{X}\|^2}{\|\bar{X} - P_2 \bar{X}\|^2} \right)^{\frac{n}{2}} = \left(\frac{\|\bar{X} - P_1 \bar{X}\|^2}{\|\bar{X} - P_1 \bar{X}\|^2 + \|P_1 \bar{X} - P_2 \bar{X}\|^2} \right)^{\frac{n}{2}} \\ &\quad \text{Pythagoras} \\ &= \left(\frac{1}{1 + \frac{\|P_1 \bar{X} - P_2 \bar{X}\|^2}{\|\bar{X} - P_1 \bar{X}\|^2}} \right)^{n/2} \end{aligned}$$

Da $P_2 \bar{X} = \sum_{i=1}^{d_2} \underline{e}_i \underline{e}_i^T \bar{X}$ ser at denne er uafh.

af SSD_{12}, SSD_2 ; da $\bar{X}_1 + \bar{X}_2, \frac{\bar{X}_1}{\bar{X}_1 + \bar{X}_2}$ er uafh.

for $\bar{X}_1 \sim \Gamma(\alpha_1, \beta), \bar{X}_2 \sim \Gamma(\alpha_2, \beta)$ uafh. og $SSD_{12} \perp SSD_2$

følger at $\hat{\mu}, \hat{\sigma}_2^2, Q_{1 \rightarrow 2}$ er uafh. under M_2

Små værdier af Q er kritiske, hvilket er ækvivalent med store værdier af $\frac{SSD_{12}/(d_1 - d_2)}{SSD_1/(n - d_1)} \sim F(d_1 - d_2, n - d_1)$

idet $SSD_{12}/(d_1 - d_2) \perp SSD_1/(n - d_1)$ og disse er hh. $\sigma^2 \chi^2(d_1 - d_2)/(d_1 - d_2), \sigma^2 \chi^2(n - d_1)/(n - d_1)$ fordelt.

□

Den todimensionale normalfordeling

Def.

$(X, Y) \sim N_2((\mu_1, \mu_2), \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix})$, hvis tetheden er

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)} \left(\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 \right)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \frac{1}{\sqrt{2\pi\sigma_2^2(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)} \left(\left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 \right)}$$

$$(x) = (-1 + \rho^2 + 1) \left(\frac{x-\mu_1}{\sigma_1} \right)^2 - 2\rho \left(\frac{x-\mu_1}{\sigma_1} \right) \left(\frac{y-\mu_2}{\sigma_2} \right) + \left(\frac{y-\mu_2}{\sigma_2} \right)^2$$

$$= \left(-\rho \left(\frac{x-\mu_1}{\sigma_1} \right) + \left(\frac{y-\mu_2}{\sigma_2} \right) \right)^2 = \left(\frac{y}{\sigma_2} - \left(\frac{\mu_2}{\sigma_2} - \frac{\rho\mu_1}{\sigma_1} + \frac{\rho}{\sigma_1} x \right) \right)^2$$

$$= f_{N(\mu_1, \sigma_1^2)}(x) \frac{1}{\sqrt{2\pi\sigma_2^2(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)\sigma_2^2} \left(y - \left(\mu_2 - \frac{\rho\mu_1\sigma_2}{\sigma_1} + \frac{\rho\sigma_2}{\sigma_1} x \right) \right)^2}$$

$$= f_{N(\mu_1, \sigma_1^2)}(x) f_{N(\alpha + \beta x, \omega^2)}(y)$$

$$\alpha = \mu_2 - \frac{\rho\mu_1\sigma_2}{\sigma_1}, \quad \beta = \rho \frac{\sigma_2}{\sigma_1}, \quad \omega^2 = (1-\rho^2)\sigma_2^2$$

$(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ varier i $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \times]-1, 1[$

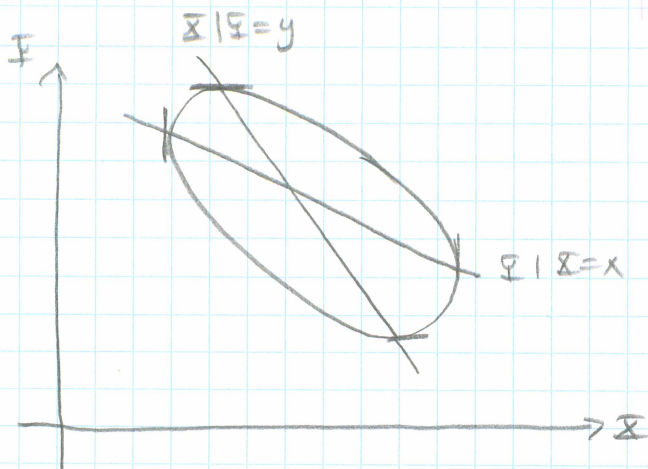
$(\mu_1, \sigma_1^2, \alpha, \beta, \omega^2)$ varier i $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$

Dvs. $(Y | X=x) \sim N(\alpha + \beta x, \omega^2) \quad \parallel \quad X \sim N(\mu_1, \sigma_1^2)$

$$L(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho) = L(\mu_1, \sigma_1^2) L(\alpha, \beta, \omega^2)$$

$$\hat{\mu}_1 = \bar{x}, \quad \sigma_1^2 = \frac{SSD_x}{n-1}, \quad \hat{\beta} = \frac{SPD_{xy}}{SSD_x} = \hat{\rho} \frac{\hat{\sigma}_2}{\hat{\sigma}_1} \implies \hat{\rho} = \frac{SPD_{xy}}{\sqrt{SSD_x SSD_y}}$$

$$\hat{\mu}_2 = \bar{y}, \quad \sigma_2^2 = \frac{SSD_y}{n-1}$$



Teste $\rho=0$: t-test (under hypotese ρ uafh. mellem est.)

Teste $\mu=\mu_0$ Hotellings T^2 -test

Teste $\mu_1=\mu_2$ (ens covariansmatricer) Hotellings T^2 test

Modelkontrol:

Teste at marginalt $\sim$ normalfordelt (frihildiagram Goodness of fit), (teste regressionslinier - est. teste sammenfald)

$$Z = \frac{1}{1-\rho^2} \left(\left(\frac{\bar{X}-\mu_1}{\sigma_1} \right)^2 - 2\rho \left(\frac{\bar{X}-\mu_1}{\sigma_1} \right) \left(\frac{\bar{Y}-\mu_2}{\sigma_2} \right) + \left(\frac{\bar{Y}-\mu_2}{\sigma_2} \right)^2 \right) \sim \chi^2(2)$$

$$f_{\chi^2(2)}(z) = \frac{1}{2} e^{-\frac{1}{2}z} \quad F_{\chi^2(2)}(z) = 1 - e^{-\frac{1}{2}z} \quad \text{dvs. man kan}$$

let tegne et frihildiagram.

Plottet en 95% konturellipse ud fra estimaterne i et scatter diagram

Oftest er den todim. normalford. ikke aktuel

- regression
- korrelationen udtrykt for en anden sammenhæng

Flerdim til: multiple korrelationskoef. - partielle kovarians kendes

$$\text{for } \bar{X}_{(1)} | \bar{X}_{(2)} = x_{(2)} \quad ; \quad \Sigma_{11} = \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

Lineære normale modell i et ortogonalt design.

Antag vi har et ortogonalt Design D
og den eneste tilfældige faktor er I , og
 T betegner mængde af systematiske faktorer

Da gælder $\underline{X} \sim N_n(\mu, \sigma^2 I_n)$, $n = |I|$

$$\mu \in L = \sum_{T \in T} L_T, \quad L_T = \text{col}(\underline{X}_T), \quad \mu = \sum_{T \in T} \underline{X}_T \alpha^T$$

Dette er jvf. teorien i kap. 4 en lin. normal model

Jvf. dekompositionssætn. kan vi skrive $\mathbb{R}^n = \bigoplus_{F \in \mathcal{D}} V_F$ entydigt

således at $L_F = \bigoplus_{G \in F} V_G$, idet Q_F betegner den
ortogonale projektion ned på V_F gælder $P_F = \sum_{G \in F} Q_G$

$$SSD_G = \|Q_G \underline{X}\|^2 \quad (\text{husk } SSD_G = SS_G - \sum_{H \neq G} SSD_H)$$

$$\text{og } \dim V_G = d_G = |G| - \sum_{H \neq G} d_H$$

Jvf. teorien om lin. normale modeller er

$$\hat{\mu} = P_L \underline{X} = \left(\sum_{T \in T} Q_T \right) \underline{X} = \sum_{T \in T} Q_T \underline{X}$$

$$\|I_n - P_L \underline{X}\|^2 = \sum_{G \notin T} \|Q_G \underline{X}\|^2 = \sum_{G \notin T} SSD_G$$

$$\sigma^2 \leftarrow S^2 = \frac{\|I_n - P_L \underline{X}\|^2}{n - \dim L} = \frac{\sum_{G \notin T} SSD_G}{\dim L} = \frac{\sum_{G \notin T} SSD_G}{\sum_{G \notin T} d_G}$$

og for $L_1 \subseteq L$, $L_1 = \sum_{T \in T_1} L_T$, $T_1 \subset T$

er F testet for reduktionen til T_1

$$F = \frac{\|P_L \bar{X} - P_{L_1} \bar{X}\|^2 / (\dim L - \dim L_1)}{\|\bar{X} - P_L \bar{X}\|^2 / (n - \dim L)}$$

da $\|\bar{X} - P_{L_1} \bar{X}\|^2 = \|\bar{X} - P_L \bar{X}\|^2 + \|P_L \bar{X} - P_{L_1} \bar{X}\|^2$ fås

idet $\|\bar{X} - P_L \bar{X}\|^2 = \sum_{G \notin T_1} SSD_G$, $n - \dim L_1 = \dim L_T = \sum_{G \notin T_1} d_G$

$$F = \frac{\sum_{G \notin T_1} SSD_G / \sum_{G \notin T_1} d_G}{\sum_{G \notin T} SSD_G / \sum_{G \notin T} d_G}$$

som skal vurderes i en $F(\dim L - \dim L_1, n - \dim L)$ ford

$$\varepsilon(x) = 1 - F_{F(.,.)}(F)$$

Sætning:

Lad $F \in T$ være fast, f_1, f_2 to niveauer

$\alpha_{f_1}^F - \alpha_{f_2}^F$ er estimbar (dvs. en fkt. af μ)

hvis $\forall T \in T \setminus \{F\}$: f_1, f_2 på samme niveau af $T \cap F$

Konsekvens $\alpha_{f_1}^F - \alpha_{f_2}^F = \bar{X}_{f_1} - \bar{X}_{f_2}$ og et konfidensint. er

$$\bar{X}_{f_1} - \bar{X}_{f_2} \pm t_{\frac{1}{2}}(\sum_{G \notin T} d_G) \sqrt{s^2 (\frac{1}{n_{f_1}} + \frac{1}{n_{f_2}})}$$

(I varianskomponentmodell: $s^2 = \lambda_{B_0}^{\sum_{G \notin T} d_G}$, betingelse er at f_1, f_2 samme niveau for $B \cap F$, $\forall B \subseteq B_0$ $F \in D_{B_0}$)

Discussion: t -test, styrke, balancet test, frøgsplanlægning.
konstant interessant: hvis en faktor ikke kan testes vekk.

Variationskomponentmodeller, herunder specielt estimation

$$\mathbf{X} \sim N_n(\boldsymbol{\mu}, \Sigma)$$

$$\boldsymbol{\mu} \in L = \sum_{T \in T} L_T$$

$$\Sigma = \sum_{B \in B} \sigma_B^2 \mathbf{Z}_B \mathbf{Z}_B^*$$

similarity
matrix

$$\sigma_B^2 \in \mathbb{R}_+$$

el. udl. model $\Sigma = \sum_{B \in B} \lambda_B \mathbf{Q}_B^0$

$$\lambda_B \in \mathbb{R}_+$$

Idet vi (da vi forudsætter alle $B \in B$ balancerede) har

$$\text{sp}(\mathbf{Z}_B \mathbf{Z}_B^*, B \in B) = \varphi(\mathbf{Q}_B^0, B \in B), \quad \lambda_{B^1} = \sum_{B \in B} \sigma_B^2 n_B$$

Vi vil estimere i den udvidede model:

Da $\mathbf{Q}_{B_1}^0 \sum \mathbf{Q}_{B_2}^{0*} = 0$ v $\mathbf{Q}_B^0 \mathbf{X}$ uafh. og

$$\mathbf{Q}_B^0 \mathbf{X} \sim N_n(\mathbf{Q}_B^0 \boldsymbol{\mu}, \lambda_B \mathbf{Q}_B^0) \text{ , og } \mathbf{Q}_B^0 \boldsymbol{\mu}, \lambda_B \text{ } B \in B$$

er variationsuafh. dvs. vi kan estimere i modellen

for $\mathbf{X}$ ved at estimere i 'delmodellen' $\mathbf{Q}_B^0 \mathbf{X}, B \in B$

som er lin. normale modeller

$$\hat{\mathbf{Q}}_B^0 \boldsymbol{\mu} = P_L \mathbf{Q}_B^0 \mathbf{X}$$

$$P_L = \sum_{T \in T^*} \mathbf{Q}_T \text{ , } \mathbf{Q}_B^0 = \sum_{F \in D_B} \mathbf{Q}_F$$

Dvs. $\hat{\boldsymbol{\mu}} = P_L \mathbf{X} = \sum_{T \in T^*} \mathbf{Q}_T \mathbf{X}$

$$\hat{\lambda}_B = \frac{\|\mathbf{Q}_B^0 \mathbf{X} - P_L \mathbf{Q}_B^0 \mathbf{X}\|^2}{\dim V_B^0 - \dim V_B^0 \cap L} = \frac{\sum_{F \in D_B \setminus T^*} \|\mathbf{Q}_F \mathbf{X}\|^2}{\sum_{F \in D_B \setminus T^*} d_F}$$

$$= \frac{\sum_{F \in D_B \setminus T^*} \text{SSD}_F}{\sum_{F \in D_B \setminus T^*} d_F} \text{ } \text{bn. } B \in D_B \setminus T^* \text{ pg. 1 (T \& B)}$$

$$\hat{\mu} = P_L \bar{X} \sim N_n(\mu, P_L \Sigma P_L') \quad \perp\!\!\!\perp$$

$$\hat{\lambda}_B = \lambda^2(f_B) \setminus (f_B) \quad , \quad f_B = \sum_{F \in \mathcal{D}_B} d_F$$

$$\text{idet } Q_B^0 \mu \perp \hat{\lambda}_B$$

Ved at benytte $\lambda_{B'} = \sum_{B \in B} \sigma_B^2 \eta_B$ estimeres $\sigma_B^2, B \in \mathcal{B}$

Hvis $\hat{\sigma}_B^2 \leq 0$ tolkes dette at med alle andre faktorer
niveauer fæste er korrel. mellem obsr. på samme niv. af B
mindre end korrel. mellem obsr. på forskellige niveauer af B.
Oftest kan B således være uinteressant i modellen på det punkt!

Estimationen for de kanoniske varianskomponenter
optræder naturligt i forbindelse med konf. int. for
kontraster $\hat{\alpha}_{F_1}^F - \hat{\alpha}_{F_2}^F$

$$\bar{X}_{F_1} - \bar{X}_{F_2} \pm t_{1-\frac{\alpha}{2}}(f_B) \sqrt{\hat{\lambda}_B \left(\frac{1}{n_{F_1}} + \frac{1}{n_{F_2}} \right)}$$

$F \in \mathcal{D}_B$ stratum og f_1, f_2 tilhører samme niv. af $B' \wedge F$
for alle $B' \in \mathcal{B}$

→ t-test i forbindelse med at F-influ. kan testes ved
styrke, type I / II fejl faseplanlægning balancerethed

Varianskomponentmodeller herunder specielt test for reduktion i middelværdistrukturen og kovariansstrukturen

Reduktion i middelværdistrukturen:

$T_i + [B]$ reduceret model

$$L = \sum_{T \in T^*} L_T, \quad L_i = \sum_{T \in T_i^*} L_T$$

Vi antager at alle $B \in B$ er balancerende og dermed gælder

$$\underset{\text{similitudismatrix}}{\text{sp}(\mathbf{X}_B \mathbf{X}_B^*, B \in B)} = \text{sp}(\mathbf{Q}_B^0, B \in B)$$

I den udr. har vi $\mathbf{X} \sim N_n(\mu, \sum_{B \in B} \lambda_B \mathbf{Q}_B^0)$

og idet $\mathbf{Q}_{B_1}^0 (\sum_{B \in B} \lambda_B \mathbf{Q}_B^0) \mathbf{Q}_{B_2}^0 = 0$ er

modellene $\mathbf{Q}_B^0 \mathbf{X}$ uafh. og $\mathbf{Q}_B^0 \mathbf{X} \sim N_n(\mathbf{Q}_B^0 \mu, \lambda_B \mathbf{Q}_B^0)$

Likelihoodratio teststatistikken for reduktionen bliver dermed

$$Q = \prod_{B \in B} \left(\frac{\|\mathbf{Q}_B^0 \mathbf{X} - P_L \mathbf{Q}_B^0 \mathbf{X}\|}{\|\mathbf{Q}_B^0 \mathbf{X} - P_{L_i} \mathbf{Q}_B^0 \mathbf{X}\|} \right)^{\frac{d_B^0}{2}}$$

Hvis $T^* \setminus T_i^* \subseteq D_{B_0}$, $B_0 \in B$ da er

$\|\mathbf{Q}_B^0 \mathbf{X} - P_L \mathbf{Q}_B^0 \mathbf{X}\| = \|\mathbf{Q}_B^0 \mathbf{X} - P_{L_i} \mathbf{Q}_B^0 \mathbf{X}\|$, $B \in B \setminus B_0$, idet

$$\mathbf{Q}_B^0 = \sum_{F \in D_B} \mathbf{Q}_F, \quad P_L = \sum_{F \in T^*} \mathbf{Q}_F, \quad P_{L_i} = \sum_{F \in T_i^*} \mathbf{Q}_F$$

Dvs. i dette tilfælde er Q ækvivalent med

$$F = \frac{\|P_L \mathbf{Q}_{B_0}^0 \mathbf{X} - P_{L_i} \mathbf{Q}_{B_0}^0 \mathbf{X}\|^2 / (\dim L - \dim L_i)}{\|\mathbf{Q}_{B_0}^0 \mathbf{X} - P_L \mathbf{Q}_{B_0}^0 \mathbf{X}\|^2 / (n - \dim L)}$$

$$= \frac{\sum_{F \in T \setminus T_0} \|Q_F \bar{X}\|^2 / \sum_{F \in T \setminus T_0} df}{\sum_{F \in D_{B_0} \setminus T^*} \|Q_F \bar{X}\|^2 / \sum_{F \in D_{B_0} \setminus T^*} df}$$

$$\|Q_F \bar{X}\|^2 = SSD_F$$

Test for $\sigma_{B_0}^2 = 0$, $B_0 \in \mathcal{B}$ i modellen $\bar{X} \sim N_n(\mu, \sum_{B \in \mathcal{B}} \sigma_B^2 \bar{X}_B \bar{X}_B')$

Idet $\lambda_{B_0} = \sum_{B_0 \in \mathcal{B}} n_B \sigma_B^2$ gælder $\lambda_{B_0} = n_{B_0} \sigma_{B_0}^2 + \lambda_{B'}$

hvor $B' = \bigwedge_{B: B_0 \not\in B} B \neq B_0$ (Vi vil antage at

vi reducerer til en varianskomponentmodel)

hypotesen $\sigma_{B_0}^2 = 0$ svarer altså til hypotesen

$$\lambda_{B_0} = \lambda_{B'}$$

Som oftest er vi kun interesseret i $\sigma_{B_0}^2 \geq 0$, der

store verdier af $\frac{\hat{\lambda}_{B_0}}{\hat{\lambda}_{B'}}$ er kritiske

Da $\hat{\lambda}_B \sim \chi^2(df_B)$, $df_B = \sum_{F \in D_B \setminus T^*} df$ er det

relevante test et F test $EC(X) = 1 - F_{F(df_{B_0}, df_{B'})}(F)$

Estimation af kontrast: Hvis f_1, f_2 (to ud af F) er $F \in D_{B_0}$

pi samme niv af BAF, $B \neq B_0$ er $\alpha_{f_1}^F - \alpha_{f_2}^F$ estimeret

af konf. int. $\bar{X}_{f_1} - \bar{X}_{f_2} \pm t_{1-\frac{\alpha}{2}}(df_{B_0}) \sqrt{\hat{\lambda}_{B_0} (\frac{1}{n_{f_1}} + \frac{1}{n_{f_2}})}$

Discussion Hvis F ikke kan testes ved

α niv., Type I, II fejl, styrke, balanceethed

Estimation og/eller test for reduktion i
middelværdistrukturen i lineære normale modeller
i det flerdimensionale tilfælde.

Sætning:

To lin. norm. modeller

$$M_1: \mathbf{X} \sim N_{n \times p}(\mathbf{M}, \mathbf{I}_n \otimes \Sigma) \quad , \quad \mathbf{P}_1 \mathbf{M} = \mathbf{M}$$

$$M_2: \mathbf{X} \sim N_{n \times p}(\mathbf{M}, \mathbf{I}_n \otimes \Sigma) \quad , \quad \mathbf{P}_2 \mathbf{M} = \mathbf{M}$$

$\mathbf{P}_1$ ortogonale projektion ned på L_1 (lin. underrum af $\mathbb{R}^n$, $\dim d_1$)

$\mathbf{P}_2$ ————— L_2 (lin. underrum af $\mathbb{R}^n$, $\dim d_2$)

$$SSD_1 = ((\mathbf{I}_n - \mathbf{P}_1)\mathbf{X})^* ((\mathbf{I}_n - \mathbf{P}_1)\mathbf{X}) = \mathbf{X}^* (\mathbf{I}_n - \mathbf{P}_1) \mathbf{X}$$

$$SSD_2 = \mathbf{X}^* (\mathbf{I}_n - \mathbf{P}_2) \mathbf{X}$$

$$SSD_{12} = \mathbf{X}^* (\mathbf{P}_1 - \mathbf{P}_2) \mathbf{X}$$

Antag Σ er regulær, $n - d_1 \geq p$, da gælder

I M_1 er maks. likelihood est. $\hat{\mathbf{M}} = \mathbf{P}_1 \mathbf{X}$, $\hat{\Sigma} = \frac{1}{n} SSD_1$

$$S_i^2 = \frac{1}{n - d_1} SSD_i \sim W_p\left(\frac{1}{n - d_1} \Sigma, n - d_1\right)$$

$$Q_{1 \rightarrow 2}^{\frac{2}{n}} = \frac{|SSD_1|}{|SSD_1 + SSD_{12}|} = B_1 \cdots B_{d_1 - d_2} \quad (\text{afh. af } \mathbf{M}, \Sigma)$$

Und M_i :

$$L(M, \Sigma) = \left(\frac{1}{2\pi}\right)^{\frac{np}{2}} \left(\frac{1}{|\Sigma|}\right)^{\frac{n}{2}} e^{-\frac{1}{2} \sum_{i=1}^n (x_i - M_i)^* \Sigma^{-1} (x_i - M_i)}$$

$$\sum_{i=1}^n (x_i - M_i)^* \Sigma^{-1} (x_i - M_i) = \text{tr}[(X - M) \Sigma^{-1} (X - M)^*]$$

$$= \text{tr}[\Sigma^{-1} (X - M)^* (X - M)] \quad (*)$$

$$(X - M) = (I_n - P_i)X + (P_i X - M)$$

$\Downarrow$

$$(X - M)^* (X - M) = X^* (I_n - P_i) X + X^* (I_n - P_i) (P_i X - M)$$

$$+ (P_i X - M)^* (I_n - P_i) X + (P_i X - M)^* (P_i X - M)$$

$$\stackrel{P_i M = M}{=} X^* (I_n - P_i) X + (P_i X - M)^* (P_i X - M)$$

$$\geq X^* (I_n - P_i) X \quad = \text{når } M = P_i X$$

$$\text{Dus. } \hat{M} = P_i X$$

$$\ell(\hat{M}, \Sigma) = -\frac{np}{2} \ln(2\pi) - \frac{n}{2} \ln(|\Sigma|) - \frac{1}{2} \text{tr}[\Sigma^{-1} X^* (I_n - P_i) X]$$

SSD_i er pos. definit (p.d.) med ss. 1 hvis $n - d_i \geq p$

Σ^{-1} er pos. definit (Σ^{-1} symmetrisk og invertibel) ved et

diagonaliserings argument $\exists \Sigma^{-\frac{1}{2}}$ invertibel så $\Sigma^{-1} = \Sigma^{-\frac{1}{2}} \Sigma^{-\frac{1}{2}*}$

og dermed ses, at $\frac{1}{n} \Sigma^{-\frac{1}{2}} \text{SSD}_i \Sigma^{-\frac{1}{2}*}$ er p.d. (invertibel symmetrisk) med egenverdier $\lambda_1, \dots, \lambda_p$

$$\ell(\hat{\mu}, \Sigma) = -\frac{np}{2} \ln(2\pi) + \frac{n}{2} \ln(|\Sigma^{-1}| |\text{SSD}_i| / n) - \frac{n}{2} \ln(|\text{SSD}_i| / n)$$

$$- \frac{n}{2} \text{tr}[\Sigma^{-1} \text{SSD}_i / n]$$

$$= -\frac{np}{2} \ln(2\pi) + \frac{n}{2} (\ln(\frac{1}{n} |\Sigma^{-\frac{1}{2}} \text{SSD}_i \Sigma^{-\frac{1}{2}*}|) - \text{tr}[\frac{1}{n} \Sigma^{-\frac{1}{2}} \text{SSD}_i \Sigma^{-\frac{1}{2}*}]) - \frac{n}{2} \ln(\frac{|\text{SSD}_i|}{n})$$

$$= -\frac{np}{2} \ln(2\pi) + \frac{n}{2} \left(\sum_{i=1}^p \ln(\lambda_i) - \lambda_i \right) - \frac{n}{2} \ln(|\text{SSD}_i| / n)$$

$$\text{Max}_i \lambda_i = 1 \quad i=1, \dots, p$$

$$\text{Dvs. for } \frac{1}{n} \Sigma^{-\frac{1}{2}} \text{SSD}_i \Sigma^{-\frac{1}{2}} = I_p \Rightarrow \Sigma = \text{SSD}_i / n$$

$$\text{Altså } \hat{\Sigma} = \text{SSD}_i / n$$

$$Q_{1 \rightarrow 2}^{\frac{2}{n}} = \left(\frac{L(P_2 \mathcal{X}, \text{SSD}_2 / n)}{L(P_1 \mathcal{X}, \text{SSD}_1 / n)} \right)^{\frac{2}{n}}$$

$$= \frac{|\text{SSD}_1|}{|\text{SSD}_2|} = \frac{|\text{SSD}_1|}{|\text{SSD}_1 + \text{SSD}_2|}$$

At $\text{SSD}_i \sim W(\Sigma, n-d_i)$ og resultatet vedr. $Q^{\frac{2}{n}}$ find
følge som i $p=1$ ved at betragte en passende
ortonormal basis for $\mathbb{R}^n$ og benytte at
 $\hat{z}^T \hat{z}$ er $W_p(\Sigma, f)$, hvis $z \sim N_{n \times p}(0, I_n \otimes \Sigma)$

Wishart fordelingen spiller rollen som χ^2 fordelingen i tilfældet
 $p=1$

□

Den negative binomial fordeling

$K \in (0, \infty)$, $\theta \in (0, 1)$ tætheden for $b^-(K, \theta)$ er
 $b^-(x; K, \theta) = \binom{K+x-1}{x} \theta^x (1-\theta)^K$, $x \in \mathbb{N}_0$

$$\binom{K+x-1}{x} = \frac{(K+x-1)(K+x-2) \cdots (K+1)K}{x!}$$

$$p := \frac{\theta}{1-\theta}, \quad Q := \frac{1}{1-\theta}$$

Da gælder for $X_n \sim \text{bi}(n, p)$, $\mathbb{E}_K \sim b^-(K, \theta)$, $Z \sim \text{po}(\lambda)$

$$\mathbb{E}[X] = n \cdot p, \quad \text{Var}(X_n) = n \cdot p \cdot (1-p) = n \cdot p \cdot q$$

$$\mathbb{E}[X_K] = K \cdot p, \quad \text{Var}(X_K) = K p Q$$

Hvis $p \rightarrow 0$, $n \rightarrow \infty$, $np \rightarrow \lambda$: $X_n \Rightarrow Z$

$p \rightarrow 0$, $K \rightarrow \infty$, $Kp \rightarrow \lambda$: $\mathbb{E}_K \Rightarrow Z$

$$\frac{\text{Var } X_n}{\mathbb{E} X_n} = q < 1 = \frac{\text{Var } Z}{\mathbb{E} Z} < \frac{\text{Var } \mathbb{E}_K}{\mathbb{E} \mathbb{E}_K} = Q$$

Hvis data er poissonfordelte gælder (Fisher's dispersion index)

$$t = \frac{s^2}{\bar{x}} \in [\chi^2_{0,025} (n-1)/(n-1); \chi^2_{0,975} (n-1)/(n-1)]$$

$$(n \geq 15 \text{ eller } \bar{x} \geq 5) \quad (s^2 = \frac{1}{n-1} (USS - \frac{s^2}{n}))$$

Hvis t ligger under intervallet har vi underspredning
og måske kan binomialfordelingen beskrive data
hvis t ligger over intervallet har vi overspredning
og den negative binomialfordeling er kandidat

$$(\text{ifly i opg. har vi } b(0; n, p) < \text{po}(0; \lambda) < b^-(0; K, \theta))$$

Eksempel: Tilfældige arbejder på granatfabrik

Poisson derer ikke (Fisher's dispersion index)

Men hvis vi antager tiden delt op i n små intervaller og $p_{ij} = p_i$ er ss. for at i'te arbejder er udsat for ulykke i j'te interval og der er uafh. mellem hændelser på forskellige intervaller samt at hændelserne er uafh. arbejder imellem, da er $(X_i \# \text{uheld i'te arbejder})$

$(X_i | \lambda = \lambda_i) \sim \text{po}(\lambda_i)$ (λ_i : intensiteten af uheld)

Hvis $\lambda \sim \Gamma(K, \beta)$ fås

$$P(X_i = x) = \int_0^{\infty} e^{-\lambda_i} \frac{\lambda_i^x}{x!} \Gamma(K)^{-1} \beta^K \lambda_i^{K-1} e^{-\beta \lambda_i} d\lambda_i$$

$$= \frac{\beta^K}{x! \Gamma(K)} \int_0^{\infty} \lambda_i^{x+K-1} e^{-(\beta+1)\lambda_i} d\lambda_i$$

$$= \frac{\beta^K}{x! \Gamma(K)} \Gamma(K+x) (\beta+1)^{-(K+x)} = \binom{K+x-1}{x} \left(\frac{1}{1+\beta}\right)^x \left(\frac{\beta}{1+\beta}\right)^K$$

$$\text{Dvs. } X_i \sim b\left(K, \frac{1}{1+\beta}\right) = \text{po}(\lambda_i) \underset{\substack{\uparrow \\ \text{mixture fordeling}}}{\lambda_i} \Gamma(K, \beta)$$

Hvis $K=1$ har vi den geometriske fordeling

$$L(K, \theta) = \prod_{i=1}^n \binom{K+x_i-1}{x_i} \theta^{x_i} (1-\theta)^K$$

$$= \theta^{x \cdot} (1-\theta)^{nK} \prod_{i=1}^n \binom{K+x_i-1}{x_i}$$

Der findes ikke et lukket udtryk for $\hat{K}, \hat{\theta}$, man udregner momentestimator $\bar{K}, \bar{\theta}$ ud fra funktion $E_{(K, \theta)}[X] = \bar{K} \frac{\bar{\theta}}{1-\bar{\theta}} = \bar{X}$.

$$\text{Var}_{(K, \theta)}(X) = \bar{K} \frac{\bar{\theta}}{1-\bar{\theta}} \frac{1}{1-\bar{\theta}} = s^2$$

$$\Rightarrow (1-\bar{\theta}) = \frac{\bar{X}}{s^2} \Rightarrow \bar{\theta} = 1 - \frac{\bar{X}}{s^2} = 1 - \frac{1}{t}$$

$$\bar{K} = \bar{X} \cdot (1-\bar{\theta}) \bar{\theta}^{-1} = \bar{X} \cdot \frac{\bar{X}}{s^2} \left(1 - \frac{\bar{X}}{s^2}\right)^{-1} = \bar{X} \cdot \left(\frac{1}{t-1}\right)$$

og bruger derefter Newton-Raphson.