

3.1 Integraler af positive funktioner

Lad $(\Omega, \mathcal{F}, \mu) \in \mathcal{M}$, $f \in \bar{M}(\Omega, \mathcal{F})$

$\bar{M}(\Omega, \mathcal{F})$ betegner mængden af alle $\mathcal{F}$ -målbar fkt. fra Ω ind i $\bar{\mathbb{R}}$

Antag, at f er positiv, dvs $f(\Omega) \subseteq [0, \infty]$

Definer:

$$\forall n \geq 1: s_n(f) := 2^{-n} \sum_{j=1}^{\infty} \mu(f > j 2^{-n})$$

$$\int f d\mu \equiv \int_{\Omega} f d\mu \equiv \int_{\Omega} f(\omega) \mu(d\omega) := \lim_{n \rightarrow \infty} s_n(f)$$

Påstand: $\int f d\mu$ er veldefineret (3.1.3)

Bevis:

Vi skal vise, at $\lim_{n \rightarrow \infty} s_n(f) = a$, hvor $a \in [0, \infty]$

$$A_j^n := \{f > j 2^{-n}\} \quad ; n, j \geq 1$$

$$A_j^n = \{f > 2j 2^{-(n+1)}\} = A_{2j}^{n+1} \subseteq \{f > (2j-1) 2^{-(n+1)}\} = A_{2j-1}^{n+1}$$

$$s_n(f) = 2^{-n} \sum_{j=1}^{\infty} \mu(A_j^n) \stackrel{1.3.2}{\leq} 2^{-n} \sum_{j=1}^{\infty} \frac{1}{2} (\underbrace{\mu(A_{2j}^{n+1})}_{\text{lige pos. heltal}} + \underbrace{\mu(A_{2j-1}^{n+1})}_{\text{ulige neg. heltal}}) = 2^{-(n+1)} \sum_{j=1}^{\infty} \mu(A_j^{n+1}) = s_{n+1}(f)$$

Dvs. $\{s_n(f)\}_{n \geq 1}$ er en voksende følge af positive tal, dermed gælder:

$$\exists a \in [0, \infty] : \lim_{n \rightarrow \infty} s_n(f) = a$$

□

Pastand:

$$f, g, f_n, n \geq 1 : \Omega \rightarrow \overline{\mathbb{R}}_+ \quad (\mathcal{F}\text{-m\u00e4\u00dflige})$$

$$(3.1.4) : f \leq g \text{ } \mu\text{-n.o.} \Rightarrow s_n(f) \leq s_n(g), n \geq 1 \wedge \int f d\mu \leq \int g d\mu$$

$$(3.1.5) : s_0(f) < \infty \Rightarrow f < \infty \text{ } \mu\text{-n.o.}$$

$$(3.1.6) : s_n(f) + s_n(g) \leq s_n(f+g) \leq s_n(f) + s_n(g) + 2^{-n} \cdot \mu(f+g > 2^{-n}), n$$

$$(3.1.7) : \int f d\mu < \infty \Rightarrow \lim_{n \rightarrow \infty} 2^{-n} \cdot \mu(f > 2^{-n}) = 0$$

$$(3.1.8) : \int (f+g) d\mu = \int f d\mu + \int g d\mu$$

$$(3.1.9) : f_n \uparrow f \Rightarrow \int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$$

$$(3.1.10) : \int (af) d\mu = a \int f d\mu, a \in \overline{\mathbb{R}}_+$$

Beweis:

3.1.4:

$$f \leq g \text{ } \mu\text{-n.o.} \Rightarrow \{f > j \cdot 2^{-n}\} \subseteq \{g > j \cdot 2^{-n}\} ; j, n \geq 1 \text{ } \mu\text{-n.o.}$$

$$\Rightarrow s_n(f) \leq s_n(g), n \geq 1 \Rightarrow \int f d\mu = \lim_{n \rightarrow \infty} s_n(f) \leq \lim_{n \rightarrow \infty} s_n(g) = \int g d\mu$$

3.1.5

$$s_0(f) = 2^{-0} \cdot \sum_{j=1}^{\infty} \mu(f > j \cdot 2^{-0}) = \sum_{j=1}^{\infty} \mu(f > j) < \infty \Rightarrow \lim_{j \rightarrow \infty} \mu(f > j) = 0$$

$$\text{og } \mu(f > 1) \leq s_0(f) < \infty$$

Af 1.4.3 f\u00f6lger, idet $\{f > j\} \downarrow \{f = \infty\}$:

$$\mu(f = \infty) = \lim_{j \rightarrow \infty} \mu(f > j) = 0,$$

Dus. $f < \infty \text{ } \mu\text{-n.o.}$

3.1.6

Lad $n \geq 1$ være givet

$$s_n(f) = \infty \Rightarrow s_n(f+g) = \infty, \text{ idet } f \leq f+g \quad (\text{og } f, g \geq 0)$$

Idet $\forall a \in \bar{\mathbb{R}}_+ : \infty + a = \infty$ følger det ønskede (idet $\mu \geq 0$)

$$s_n(f) < \infty \xRightarrow[3.1.5]{} f < \infty \mu\text{-n.o.}, \text{ dvs. } \exists N \in \mathbb{F} : \mu(N) = 0 \wedge \forall \omega \in \Omega \setminus N : f(\omega) < \infty$$

Lad $j, n \geq 1$ være vilk., definer

$$F_j^n = \{f > j \cdot 2^{-n}\}; G_j^n = \{g > j \cdot 2^{-n}\}; H_j^n = \{f+g > j \cdot 2^{-n}\}$$

$$D_0^n = \{f \leq 2^{-n}\}; D_j^n = \{j \cdot 2^{-n} < f \leq (j+1) \cdot 2^{-n}\}, j \geq 1$$

$D_0^n, D_1^n, \dots, D_{j-1}^n, F_j^n$ er disjunkte og

$$\Omega \setminus N = \bigcup_{k=0}^{\infty} D_k^n = (F_j^n \cup \bigcup_{k=0}^{j-1} D_k^n) \setminus N, j \geq 1; \Omega = F_j^n \cup \bigcup_{k=0}^{j-1} D_k^n$$

$$G_{j-k}^n \cap D_k^n = \{g > (j-k) \cdot 2^{-n}, k \cdot 2^{-n} < f \leq (k+1) \cdot 2^{-n}\}$$

$$\subseteq \{g > j \cdot 2^{-n} - f, k \cdot 2^{-n} < f \leq (k+1) \cdot 2^{-n}\} = H_j^n \cap D_k^n$$

$$\subseteq \{g > j \cdot 2^{-n} - (k+1) \cdot 2^{-n}, k \cdot 2^{-n} < f \leq (k+1) \cdot 2^{-n}\} = G_{j-k-1}^n \cap D_k^n$$

Desuden gælder (trivielt, idet $f, g \in \bar{\mathbb{R}}_+$) $F_j^n \subseteq H_j^n, G_j^n \subseteq H_j^n$

$$\begin{aligned} \mu(H_j^n) &\stackrel{F_j^n \subseteq H_j^n}{=} \mu(F_j^n) + \sum_{k=0}^{j-1} \mu(D_k^n \cap H_j^n) \stackrel{1.3.2}{\geq} \mu(F_j^n) + \sum_{k=0}^{j-1} \mu(D_k^n \cap G_{j-k}^n) \\ &\stackrel{\substack{\Omega = F_j^n \cup \bigcup_{k=0}^{j-1} D_k^n \\ + F_j^n \cap D_0^n, \dots, D_{j-1}^n \text{ disj.} \\ + \text{P.2.5, 1.3.5}}}{=} \mu(F_j^n) + \sum_{\substack{k=j-i \\ i=1, \dots, j}} \mu(D_{j-i}^n \cap G_i^n) \quad (*) \end{aligned}$$

$$\begin{aligned} \mu(H_j^n) &\stackrel{\downarrow}{=} \mu(F_{j-1}^n) + \sum_{k=0}^{j-2} \mu(D_k^n \cap H_j^n) \stackrel{1.3.2}{\leq} \mu(F_{j-1}^n) + \sum_{k=0}^{j-2} \mu(D_k^n \cap G_{j-k-1}^n) \\ &\stackrel{\substack{k=j-1-i \\ i=1, \dots, j-1}}{=} \mu(F_{j-1}^n) + \sum_{i=1}^{j-1} \mu(D_{j-1-i}^n \cap G_i^n) \quad (**) \end{aligned}$$

$$\begin{aligned}
s_n(f+g) &= 2^{-n} \sum_{j=1}^{\infty} \mu(f+g > j2^{-n}) = 2^{-n} \sum_{j=1}^{\infty} \mu(H_j^n) \\
&\stackrel{(\star)}{\geq} 2^{-n} \sum_{j=1}^{\infty} \mu(F_j^n) + 2^{-n} \sum_{j=1}^{\infty} \sum_{i=1}^j \mu(D_{j-i}^n \cap G_i^n) \\
&\stackrel{3.1.1}{=} s_n(f) + 2^{-n} \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \mu(D_{j-i}^n \cap G_i^n) \\
&= s_n(f) + 2^{-n} \sum_{i=1}^{\infty} \mu((\mathbb{R} \setminus N) \cap G_i^n) = s_n(f) + 2^{-n} \sum_{i=1}^{\infty} \mu(G_i^n) \\
&\stackrel{3.1.1}{=} s_n(f) + s_n(g)
\end{aligned}$$

$$\begin{aligned}
s_n(f+g) &\stackrel{3.1.1}{=} 2^{-n} \sum_{j=1}^{\infty} \mu(H_j^n) \stackrel{(\star\star)}{\leq} 2^{-n} \mu(H_1^n) + 2^{-n} \sum_{j=2}^{\infty} \mu(F_{j-1}^n) + 2^{-n} \sum_{j=2}^{\infty} \sum_{i=1}^{j-1} \mu(D_{j-i-1}^n \cap G_i^n) \\
&\stackrel{1.3.2}{=} 2^{-n} \mu(H_1^n) + 2^{-n} \sum_{j=1}^{\infty} \mu(F_j^n) + 2^{-n} \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \mu(D_{j-i-1}^n \cap G_i^n) \\
&= 2^{-n} \mu(f+g > 2^{-n}) + s_n(f) + 2^{-n} \sum_{i=1}^{\infty} \mu((\mathbb{R} \setminus N) \cap G_i^n) \\
&= 2^{-n} \mu(f+g > 2^{-n}) + s_n(f) + 2^{-n} \sum_{i=1}^{\infty} \mu(G_i^n) = s_n(f) + s_n(g) + 2^{-n} \mu(f+g > 2^{-n})
\end{aligned}$$

3.1.7

$$\forall n \geq 1: a_n := 2^{-n} \mu(f > 2^{-n}) \leq 2^{-n} \sum_{j=1}^{\infty} \mu(f > j2^{-n}) =: s_1(f) \leq s_n(f)$$

Desuden gælder, idet $\{s_n(f)\}_{n \geq 1}$ er en voksende følge af pos. reelle tal, der konvergerer mod $\int f d\mu$, at $\forall n \geq 1: s_n(f) \leq \int f d\mu$

Vi antager, at $\int f d\mu < \infty$ og får:

$$\forall n \geq 1: 0 \leq a_n \leq s_n(f) \leq \int f d\mu < \infty, \quad (\star)$$

Da $\{s_n(f)\}_{n \geq 1}$ er en voksende følge, der konvergerer mod $\int f d\mu$ gælder juf. def. på $\limsup_{n \rightarrow \infty} s_n(f)$, $\liminf_{n \rightarrow \infty} s_n(f)$ og P. 4.22, at

$$\liminf_{n \rightarrow \infty} s_n(f) = \limsup_{n \rightarrow \infty} s_n(f) = \lim_{n \rightarrow \infty} s_n(f) = \int f d\mu < \infty$$

Def. $\forall n, j \geq 1$: $F_j^n := \{f > j \cdot 2^{-n}\}$ (som tidl. brn: $F_j^n = F_{2j}^{n+1} \subseteq F_{2j-1}^{n+1}$ **)

$$\begin{aligned} \frac{1}{2} a_n + s_{n+1}(f) &= \frac{1}{2} \cdot 2^{-n} \cdot \mu(f > 2^{-n}) + 2^{-n-1} \sum_{j=1}^{\infty} \mu(f > j \cdot 2^{-n-1}) \\ &= 2^{-n-1} \mu(F_1^n) + 2^{-n-1} \cdot \mu(F_1^{n+1}) + 2^{-n-1} \cdot \sum_{j=2}^{\infty} \mu(F_j^{n+1}) \\ &= a_{n+1} + 2^{-n-1} \left(\mu(F_1^n) + \sum_{j=2}^{\infty} \mu(F_{2j-1}^{n+1}) + \sum_{j=1}^{\infty} \mu(F_{2j}^{n+1}) \right) \\ &\stackrel{**}{\geq} a_{n+1} + 2^{-n-1} \left(\mu(F_1^n) + \sum_{j=2}^{\infty} \mu(F_j^n) + \sum_{j=1}^{\infty} \mu(F_j^n) \right) \\ &= a_{n+1} + 2^{-n-1} \cdot (2 \cdot \sum_{j=1}^{\infty} \mu(F_j^n)) = a_{n+1} + s_n(f) \end{aligned}$$

Af (*) følger: $0 \leq \limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} s_n(f) < \infty$,
da P. 4.19 jvf. det ovenstående giver:

$$\limsup_{n \rightarrow \infty} \frac{1}{2} a_n + \limsup_{n \rightarrow \infty} s_{n+1}(f) \geq \limsup_{n \rightarrow \infty} a_{n+1} + \liminf_{n \rightarrow \infty} s_n(f)$$

og idet $\liminf_{n \rightarrow \infty} s_{n+1}(f) = \limsup_{n \rightarrow \infty} s_n(f)$ fås

$$\frac{1}{2} \limsup_{n \rightarrow \infty} a_n \geq \limsup_{n \rightarrow \infty} a_n \quad \text{og} \quad \limsup_{n \rightarrow \infty} a_n < \infty$$

hvilket betyder, at $\limsup_{n \rightarrow \infty} a_n = 0 \Rightarrow \lim_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n = 0$
 $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$ (jvf. *)

Dvs. (def. på a_n) at $\lim_{n \rightarrow \infty} 2^{-n} \cdot \mu(f > 2^{-n}) = 0$ hvis $\int f d\mu < \infty$

3.1.8

Af 3.1.6 følger

$$(*) \int f d\mu + \int g d\mu := \lim_{n \rightarrow \infty} s_n(f) + \lim_{n \rightarrow \infty} s_n(g) \leq \lim_{n \rightarrow \infty} s_n(f+g) =: \int (f+g) d\mu$$

Dvs. hvis $\int f d\mu = \infty$ v $\int g d\mu = \infty$ gælder

$$\int f d\mu + \int g d\mu = \int (f+g) d\mu$$

Antag $\int f d\mu < \infty \wedge \int g d\mu < \infty$

Af 3.1.7 følger, at $\lim_{n \rightarrow \infty} (2^{-n} \cdot \mu(f > 2^{-n}) + 2^{-n} \mu(g > 2^{-n})) = 0$

$$2^{-n} \mu(f+g > 2^{-n}) \leq 2^{-n} \mu(f > 2^{-n-1}) + 2^{-n} \mu(g > 2^{-n-1})$$

(idet $\{f+g > 2^{-n}\} \subseteq \{f > 2^{-n-1}\} \cup \{g > 2^{-n-1}\}$ + 1.3.6)

Dvs. $\lim_{n \rightarrow \infty} 2^{-n} \mu(f+g > 2^{-n}) = 0$

Af 3.1.6 følger

$$\begin{aligned} (**) \quad \int (f+g) d\mu &:= \lim_{n \rightarrow \infty} s_n(f+g) \leq \lim_{n \rightarrow \infty} s_n(f) + \lim_{n \rightarrow \infty} s_n(g) + \lim_{n \rightarrow \infty} 2^{-n} \mu(f+g > 2^{-n}) \\ &= \int f d\mu + \int g d\mu \end{aligned}$$

$$(*) + (**) \implies \int (f+g) d\mu = \int f d\mu + \int g d\mu$$

3.1.9

Antag $f_n \uparrow f$ for $n \uparrow \infty$

Hvis $\int f d\mu = 0$ gælder jvf. 3.1.4, at

$$0 \leq \int f_n d\mu \leq \int f d\mu \leq 0, \quad n \geq 1$$

Dvs. $\forall n \geq 1 : \int f_n d\mu = 0$ og dermed trivielt: $\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu$

Antag $\int f d\mu > 0$, dvs. $\exists a \in [0, \infty): a < \int f d\mu$

Af def. 3.1.2 (på $\int f d\mu$) følger, at $\exists p \geq 1: s_p(f) > a$

Af def. 3.1.1 (på $s_p(f)$) følger, at $\exists q \geq 1: 2^{-p} \cdot \sum_{j=1}^q \mu(f > j \cdot 2^{-p}) > a$

Idet $\{f_n > j \cdot 2^{-p}\} \uparrow_{n \rightarrow \infty} \{f > j \cdot 2^{-p}\}$ ($f_n \uparrow f$), hvor $p, j \geq 1$ følger af 1.4.2, at $\mu(f > j \cdot 2^{-p}) = \lim_{n \rightarrow \infty} \mu(f_n > j \cdot 2^{-p})$; $p, j \geq 1$

Dvs. $\exists k \geq 1: a < 2^{-p} \cdot \sum_{j=1}^k \mu(f_k > j \cdot 2^{-p}) \leq s_p(f_k) \leq \int f_k d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu$
3.1.4

Ved at lade a nærme sig $\int f d\mu$ ser vi, at $\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu$

3.1.10

Lad $a \in \bar{\mathbb{R}}_+$ være givet

Antag $a \in \mathbb{N}_0$

Påstand: $a \cdot \int f d\mu = \int a \cdot f d\mu$ (*)

Bevis ved induktion:

$a=0$:

$0 \cdot \int f d\mu = 0 = \int 0 d\mu = \int 0 \cdot f d\mu$ o.k.

Antag påstanden sand for $a=n$:

$(n+1) \cdot \int f d\mu = n \cdot \int f d\mu + \int f d\mu = \int n \cdot f d\mu + \int f d\mu \stackrel{\substack{\text{induktions} \\ \text{antagelse}}}{=} \int (n+1)f d\mu \stackrel{3.1.8}{=}$

Dvs. påstanden er sand for $n+1$. □

Antag $a \in \mathbb{Q}_+$

$\exists k \in \mathbb{N}, m \in \mathbb{N}_0: a = \frac{m}{k}$

Påstand: $a \cdot \int f d\mu = \int a \cdot f d\mu$ (**)

Bevis:

$m \cdot \int f d\mu \stackrel{(*)}{=} \int m f d\mu = \int k \cdot \left(\frac{m}{k}\right) \cdot f d\mu \stackrel{(**)}{=} k \int \left(\frac{m}{k}\right) \cdot f d\mu = k \cdot \int a f d\mu$

$\Rightarrow a \cdot \int f d\mu = \frac{m}{k} \cdot \int f d\mu = \frac{k}{k} \cdot \int a f d\mu = \int a f d\mu$ □

Antag $a \in \overline{\mathbb{R}}_+$

$\exists q_n, n \geq 1 \in \mathbb{Q}_+ : \lim_{n \rightarrow \infty} q_n = a, \quad \forall n \geq 1: q_{n+1} \geq q_n \quad (+)$

Def. $\forall n \geq 1: f_n := q_n \cdot f$

Af (+) folger, at $f_n \uparrow a \cdot f$, dus

$$\int (a \cdot f) d\mu \underset{3.1.9}{=} \lim_{n \rightarrow \infty} \int f_n d\mu = \lim_{n \rightarrow \infty} \int q_n f d\mu \underset{(**)}{=} \lim_{n \rightarrow \infty} (q_n \cdot \int f d\mu) = a \cdot \int f d\mu$$

□

3.2 Integraler af reelle funktioner

$(\Omega, \mathcal{F}, \mu) @ \omega$, $f \in \overline{M}(\Omega, \mathcal{F})$ (se note til 3.1)

$$P.4.1 : f = f^+ - f^- , |f| = f^+ + f^- \quad (3.2.1)$$

(Det er klart, at $f^+, f^-, |f|$ er målelige fkt. fra Ω ind i $\overline{\mathbb{R}}_+$)

μ -integral af f :

$$\int f d\mu = \int_{\Omega} f d\mu = \int_{\Omega} f(\omega) \mu(d\omega) := \int f^+ d\mu - \int f^- d\mu \quad (3.2.2)$$

Hvis $\int_{\Omega} f^+ d\mu = \int_{\Omega} f^- d\mu = \infty$ er $\int f d\mu$ ikke veldefineret

$$L(\mu) := \{f \in \overline{M}(\Omega, \mathcal{F}) \mid \int_{\Omega} f d\mu \text{ er veldf.}\} = \{f \in \overline{M}(\Omega, \mathcal{F}) \mid \int_{\Omega} f^+ d\mu < \infty \vee \int_{\Omega} f^- d\mu < \infty\} \quad (3.2.4) \quad (3.2.3)$$

Dvs. $L(\mu)$ er mængden af fkt. tilhørende $\overline{M}(\Omega, \mathcal{F})$ med veldefinerede μ -integraler.

$f \in L(\mu)$ er μ -integrabel, hvis $\int_{\Omega} f d\mu \neq \pm \infty$

$$L^1(\mu) := \{f \in L(\mu) \mid f \text{ er } \mu\text{-integrabel}\} = \{f \in L(\mu) \mid \int_{\Omega} f d\mu \neq \pm \infty\} \quad (3.2.5)$$

Juf. def. på $\int_{\Omega} f d\mu$ ses

$$L^1(\mu) = \{f \in \overline{M}(\Omega, \mathcal{F}) \mid \int_{\Omega} f^+ d\mu < \infty \wedge \int_{\Omega} f^- d\mu < \infty\} \quad (3.2.6)$$

$$\stackrel{|f| = f^+ + f^-}{=} \{f \in \overline{M}(\Omega, \mathcal{F}) \mid \int_{\Omega} |f| d\mu < \infty\} \quad (3.2.7)$$

Lad $F \in \mathcal{F}$, $f \in \overline{M}(\Omega, \mathcal{F})$ μ -integral af f over F defineres som:

$$\int_F f d\mu = \int_F f(\omega) \mu(d\omega) := \int_{\Omega} 1_F(\omega) f(\omega) \mu(d\omega); \text{ veldf. for } 1_F f \in L(\mu) \quad (3.2.8)$$

3.3 Integralens analytiske egenskaber

$(\Omega, \mathcal{F}, \mu) @$; $f, g \in L(\mu)$, $h \in \overline{\mathbb{R}}(\Omega, \mathcal{F})$

Påstand:

$$(3.3.1): \int_{\Omega} f d\mu > -\infty \wedge f \leq h \text{ } \mu\text{-n.o.} \implies h \in L(\mu)$$

$$(3.3.2): \int_{\Omega} f d\mu < \infty \wedge f \geq h \text{ } \mu\text{-n.o.} \implies h \in L(\mu)$$

$$(3.3.3): f \leq g \text{ } \mu\text{-n.o.} \implies \int_{\Omega} f d\mu \leq \int_{\Omega} g d\mu$$

$$(3.3.4): f = g \text{ } \mu\text{-n.o.} \implies \int_{\Omega} f d\mu = \int_{\Omega} g d\mu$$

$$(3.3.5): f \in L^1(\mu); a \in \mathbb{R}_+ : |h| \leq a|f| \text{ } \mu\text{-n.o.} \implies h \in L^1(\mu)$$

$$(3.3.6): \int_{\Omega} f d\mu < \infty \implies f < \infty \text{ } \mu\text{-n.o.}$$

$$(3.3.7): \int_{\Omega} f d\mu > -\infty \implies f > -\infty \text{ } \mu\text{-n.o.}$$

$$(3.3.8): \int_{\Omega} |f| d\mu = 0 \iff f = 0 \text{ } \mu\text{-n.o.}$$

$$(3.3.9): \left| \int_{\Omega} f d\mu \right| \leq \int_{\Omega} |f| d\mu$$

$$(3.3.10): L^1(\mu) \text{ er et vektorrum og } f \mapsto \int_{\Omega} f d\mu \text{ er linear p\u00e5 } L^1(\mu)$$

$$(3.3.11): \forall a \in \mathbb{R} : \int_{\Omega} (af) d\mu = a \int_{\Omega} f d\mu$$

$$(3.3.12): \int_{\Omega} (f+g) d\mu = \int_{\Omega} f d\mu + \int_{\Omega} g d\mu \quad (\text{antag, at } f+g, \int_{\Omega} f d\mu + \int_{\Omega} g d\mu \text{ er veldef.})$$

(dvs. ikke p\u00e5 formen $\infty - \infty$)

$$(3.3.13): \exists a \in \mathbb{R}_+ : 0 \leq h \leq a \implies h \cdot f \in L(\mu)$$

Beweis:

3.3.1:

$$\int_{\Omega} f d\mu > -\infty \quad \wedge \quad f \leq h \quad \mu\text{-n.o.}$$

$$\int_{\Omega} f d\mu = \int_{\Omega} \underbrace{f^+}_{\geq 0} d\mu - \int_{\Omega} \underbrace{f^-}_{\geq 0} d\mu > -\infty \implies \int_{\Omega} f^- d\mu < \infty$$

$$f \leq h \quad \mu\text{-n.o.} \xRightarrow[\substack{\text{def. } \tilde{\mu} \\ x \mapsto x^-}]{\text{3.1.4}} f^- \geq h^- \quad \mu\text{-n.o.} \xRightarrow{3.1.4} \int_{\Omega} f^- d\mu \geq \int_{\Omega} h^- d\mu$$

Dus. $\int_{\Omega} h^- d\mu < \infty$ og dermed (jvf. 3.2.3) at $h \in L(\mu)$

3.3.2:

$$\int_{\Omega} f d\mu < \infty \quad \wedge \quad f \geq h \quad \mu\text{-n.o.}$$

$$\int_{\Omega} f d\mu = \int_{\Omega} \underbrace{f^+}_{\geq 0} d\mu - \int_{\Omega} \underbrace{f^-}_{\geq 0} d\mu < \infty \implies \int_{\Omega} f^+ d\mu < \infty$$

$$f \geq h \quad \mu\text{-n.o.} \xRightarrow[\substack{\text{def. } \tilde{\mu} \\ x \mapsto x^+}]{\text{3.1.4}} f^+ \geq h^+ \quad \mu\text{-n.o.} \xRightarrow{3.1.4} \int_{\Omega} f^+ d\mu \geq \int_{\Omega} h^+ d\mu$$

Dus. $\int_{\Omega} h^+ d\mu < \infty$ og dermed (jvf. 3.2.3) at $h \in L(\mu)$

3.3.3

$$f \underset{3.2.1}{=} f^+ - f^- \leq g^+ - g^- \underset{3.2.1}{=} g \quad \mu\text{-n.o.}$$

$\Downarrow$

$$f^+ + g^- \leq g^+ + f^- \quad \mu\text{-n.o.}$$

$\Downarrow$ 3.1.4 + 3.1.8

$$\int_{\Omega} f^+ d\mu + \int_{\Omega} g^- d\mu \leq \int_{\Omega} g^+ d\mu + \int_{\Omega} f^- d\mu$$

$\Downarrow$

$$\int_{\Omega} f d\mu := \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu \leq \int_{\Omega} g^+ d\mu - \int_{\Omega} g^- d\mu =: \int_{\Omega} g d\mu$$

3.3.4

$$f = g \quad \mu\text{-n.o.} \Rightarrow f \leq g \wedge f \geq g \quad \mu\text{-n.o.}$$

$$\Rightarrow \int_{\Omega} f d\mu \leq \int_{\Omega} g d\mu \wedge \int_{\Omega} f d\mu \geq \int_{\Omega} g d\mu \Rightarrow \int_{\Omega} f d\mu = \int_{\Omega} g d\mu$$

3.3.5

$$f \in L^1(\mu), \exists a \in \mathbb{R}_+ : |h| \leq a \cdot |f| \quad \mu\text{-n.o.}$$

$$f \in L^1(\mu) \underset{3.2.7}{\iff} \int_{\Omega} |f| d\mu < \infty \underset{a \in \mathbb{R}_+}{\implies} a \cdot \int_{\Omega} |f| d\mu < \infty$$

$$|h| \leq a \cdot |f| \quad \mu\text{-n.o.} \underset{\substack{a \in \mathbb{R}_+ \\ 3.1.4}}{\implies} \int_{\Omega} |h| d\mu \leq \int_{\Omega} a \cdot |f| d\mu \underset{3.1.10}{=} a \cdot \int_{\Omega} |f| d\mu$$

Dus. $\int_{\Omega} |h| d\mu < \infty$, og dermed (husk $h \in \mathcal{M}(\Omega, \mathcal{F})$) $h \in L^1(\mu)$
 Def. 3.2.7

3.3.6:

$$\int_{\mathbb{R}} f d\mu < \infty \xRightarrow{3.2.2} \int_{\mathbb{R}} f^+ d\mu - \int_{\mathbb{R}} f^- d\mu < \infty$$

$$\Rightarrow \int_{\mathbb{R}} f^+ d\mu < \infty \xRightarrow[3.1.5]{\substack{S_0(f^+) \in S f^+ d\mu \\ \text{vgl. 3.1.3}}} f^+ < \infty \mu\text{-n.o.}$$

$$f \stackrel{3.2.1}{=} f^+ - f^-$$

Dus. $f < \infty$ μ -n.o.

3.3.7

$$\int_{\mathbb{R}} f d\mu > -\infty \xRightarrow{3.2.2} \int_{\mathbb{R}} f^+ d\mu - \int_{\mathbb{R}} f^- d\mu > -\infty$$

$$\Rightarrow \int_{\mathbb{R}} f^- d\mu < \infty \xRightarrow[3.1.5]{\substack{S_0(f^-) \in S f^- d\mu \\ \text{vgl. 3.1.3}}} f^- < \infty \mu\text{-n.o.}$$

$$f \stackrel{3.2.1}{=} f^+ - f^-$$

Dus. $f > -\infty$ μ -n.o.

3.3.8

Paßand: $\int_{\Omega} |f| d\mu = 0 \iff f = 0 \quad \mu\text{-n.o.}$

Bevis:

' $\implies$ ':

$$\int_{\Omega} |f| d\mu = 0 \xrightarrow{3.1.3} \forall n \geq 1: s_n(|f|) = 0 \implies \forall n \geq 1: \mu(|f| > 2^{-n}) = 0$$

$\{|f| > 2^{-n}\} \uparrow \{f \neq 0\}$ af 1.4.2 følger $\mu(f \neq 0) = \lim_{n \rightarrow \infty} \mu(|f| > 2^{-n}) = 0$

Dvs. $f = 0 \quad \mu\text{-n.o.}$

' $\Leftarrow$ ':

$$f = 0 \quad \mu\text{-n.o.} \xrightarrow{f = f^+ - f^-} f^+ = 0 \wedge f^- = 0 \quad \mu\text{-n.o.}$$

$$\xrightarrow{3.1.4} \int_{\Omega} f^+ d\mu = \int_{\Omega} f^- d\mu = 0$$

$$\int_{\Omega} |f| d\mu = \int_{\Omega} (f^+ + f^-) d\mu \stackrel{3.1.8}{=} \int_{\Omega} f^+ d\mu + \int_{\Omega} f^- d\mu = 0$$

3.3.9

$$\left| \int_{\Omega} f d\mu \right| \stackrel{3.2.2}{=} \left| \underbrace{\int_{\Omega} f^+ d\mu}_0 - \underbrace{\int_{\Omega} f^- d\mu}_0 \right| \leq \left| \int_{\Omega} f^+ d\mu + \int_{\Omega} f^- d\mu \right|$$

$$\stackrel{3.1.8}{=} \left| \int_{\Omega} (f^+ + f^-) d\mu \right| \stackrel{3.1.4}{=} \left| \int_{\Omega} |f| d\mu \right| = \int_{\Omega} |f| d\mu$$

3.3.10

Påstand: $L^1(\mu)$ er et vektorrum

Bevis:

Lad $u, v, w \in L^1(\mu)$; $r, s \in \mathbb{R}$ være givet

$r \cdot u, u+v \in L^1(\mu)$: (brukt bevismetoden i 3.3.11-12)

$r < 0$:

$$\int_{\mathcal{X}} r \cdot u d\mu := \int_{\mathcal{X}} (r \cdot u)^+ d\mu - \int_{\mathcal{X}} (r \cdot u)^- d\mu$$

$$= |r| \cdot \frac{1}{|r|} \cdot \left(\int_{\mathcal{X}} (r \cdot u)^+ d\mu - \int_{\mathcal{X}} (r \cdot u)^- d\mu \right)$$

$$= |r| \cdot \left(\int_{\mathcal{X}} \left(\frac{r}{|r|} u \right)^+ d\mu - \int_{\mathcal{X}} \left(\frac{r}{|r|} u \right)^- d\mu \right)$$

$$\frac{1}{|r|} (ru)^+ = \left(\frac{r}{|r|} u \right)^+$$

$$\frac{1}{|r|} (ru)^- = \left(\frac{r}{|r|} u \right)^-$$

3.1.10

$$\stackrel{r < 0}{=} |r| \cdot \left(\int_{\mathcal{X}} (-u)^+ d\mu - \int_{\mathcal{X}} (-u)^- d\mu \right) = |r| \cdot \left(\int_{\mathcal{X}} u^- d\mu - \int_{\mathcal{X}} u^+ d\mu \right)$$

$$\stackrel{3.2.2}{=} |r| \cdot \left(- \int_{\mathcal{X}} u d\mu \right) \stackrel{r < 0}{=} r \int_{\mathcal{X}} u d\mu$$

$r = 0$:

$$\int_{\mathcal{X}} r \cdot u d\mu = \int_{\mathcal{X}} 0 d\mu = 0 = 0 \cdot \int_{\mathcal{X}} u d\mu = r \cdot \int_{\mathcal{X}} u d\mu$$

$r > 0$:

$$\int_{\mathcal{X}} r \cdot u d\mu = \int_{\mathcal{X}} (ru)^+ d\mu - \int_{\mathcal{X}} (ru)^- d\mu = |r| \cdot \frac{1}{|r|} \left(\int_{\mathcal{X}} (ru)^+ d\mu - \int_{\mathcal{X}} (ru)^- d\mu \right)$$

$$\stackrel{r > 0}{=} |r| \cdot \left(\int_{\mathcal{X}} \left(\frac{r}{|r|} u \right)^+ d\mu - \int_{\mathcal{X}} \left(\frac{r}{|r|} u \right)^- d\mu \right) = r \left(\int_{\mathcal{X}} u^+ d\mu - \int_{\mathcal{X}} u^- d\mu \right)$$

$$\frac{1}{|r|} (ru)^+ = \left(\frac{r}{|r|} u \right)^+$$

$$\frac{1}{|r|} (ru)^- = \left(\frac{r}{|r|} u \right)^-$$

3.1.10

$$= r \cdot \int_{\mathcal{X}} u d\mu$$

$$u+v = (u+v)^+ - (u+v)^- = u^+ - u^- + v^+ - v^-$$

$\Downarrow$

$$(u+v)^+ + u^- + v^- = (u+v)^- + u^+ + v^+$$

$\Downarrow$ 3.1.8

$$\int_{\mathbb{R}} (u+v)^+ d\mu + \int_{\mathbb{R}} u^- d\mu + \int_{\mathbb{R}} v^- d\mu = \int_{\mathbb{R}} (u+v)^- d\mu + \int_{\mathbb{R}} u^+ d\mu + \int_{\mathbb{R}} v^+ d\mu$$

$\Downarrow$ (husk $u, v \in L^1(\mu)$)

$$\int_{\mathbb{R}} (u+v) d\mu := \int_{\mathbb{R}} (u+v)^+ d\mu - \int_{\mathbb{R}} (u+v)^- d\mu = \int_{\mathbb{R}} u^+ d\mu - \int_{\mathbb{R}} u^- d\mu + \int_{\mathbb{R}} v^+ d\mu - \int_{\mathbb{R}} v^- d\mu =: \int_{\mathbb{R}} u d\mu + \int_{\mathbb{R}} v d\mu$$

At A1-A4, S1-S4 i 'Linear Algebra, Frølich and Beauregard' s. 180-181 er opfyldt følger næmt:

□

3.3.11 (dette bevis kan med fordel benyttes i 3.3.10)

Lad $a \in \mathbb{R}$ være vilkårlig

Påstand: $a \cdot \int_{\mathbb{R}} f d\mu = \int_{\mathbb{R}} a f d\mu$

Bevis:

$$(af)^+ = \begin{cases} a f^+, & a \geq 0 \\ -a f^-, & a \leq 0 \end{cases} ; \quad (af)^- = \begin{cases} a f^-, & a \geq 0 \\ -a f^+, & a \leq 0 \end{cases}$$

$$a \cdot \int_{\mathbb{R}} f d\mu = a \cdot \left(\int_{\mathbb{R}} f^+ d\mu - \int_{\mathbb{R}} f^- d\mu \right)$$

$$= \begin{cases} \int_{\mathbb{R}} a \cdot f^+ d\mu - \int_{\mathbb{R}} a \cdot f^- d\mu = \int_{\mathbb{R}} (af)^+ d\mu - \int_{\mathbb{R}} (af)^- d\mu =: \int_{\mathbb{R}} a f d\mu, & a \geq 0 \\ \int_{\mathbb{R}} (-a) \cdot f^+ d\mu + \int_{\mathbb{R}} (-a) \cdot f^- d\mu = - \int_{\mathbb{R}} (af)^- d\mu + \int_{\mathbb{R}} (af)^+ d\mu =: \int_{\mathbb{R}} a f d\mu, & a \leq 0 \end{cases}$$

□

3.3.12

Påstand: $\int_{\mathcal{X}} f d\mu + \int_{\mathcal{X}} g d\mu = \int_{\mathcal{X}} (f+g) d\mu$ (antager $f+g$, $\int_{\mathcal{X}} f d\mu + \int_{\mathcal{X}} g d\mu$ ikke er på formen ∞)

Bevis:

$$f+g = (f+g)^+ - (f+g)^- = f^+g^+ - f^-g^- \quad (*)$$

$\Downarrow$

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

$\Downarrow$ 3.1.8

$$\int_{\mathcal{X}} (f+g)^+ d\mu + \int_{\mathcal{X}} f^- d\mu + \int_{\mathcal{X}} g^- d\mu = \int_{\mathcal{X}} (f+g)^- d\mu + \int_{\mathcal{X}} f^+ d\mu + \int_{\mathcal{X}} g^+ d\mu$$

$\Downarrow$

$$\int_{\mathcal{X}} (f+g)^+ d\mu - \int_{\mathcal{X}} (f+g)^- d\mu = \int_{\mathcal{X}} f^+ d\mu - \int_{\mathcal{X}} f^- d\mu + \int_{\mathcal{X}} g^+ d\mu - \int_{\mathcal{X}} g^- d\mu \quad (**)$$

$\Downarrow$

$$\int_{\mathcal{X}} (f+g) d\mu = \int_{\mathcal{X}} f d\mu + \int_{\mathcal{X}} g d\mu$$

□

3.3.13

Påstand: $\exists a \in \mathbb{R}_+ : 0 \leq h \leq a \Rightarrow h \cdot f \in L(\mu)$ (klod at $h \cdot f \in \overline{M}(\mathcal{X}, \mathcal{F})$ jf. 1.43.5)

Bevis:

$\exists a \in \mathbb{R}_+ : 0 \leq h \leq a, h \in \overline{M}(\mathcal{X}, \mathcal{F}) ; f \in L(\mu) : \int_{\mathcal{X}} f^+ d\mu < \infty \vee \int_{\mathcal{X}} f^- d\mu < \infty$ (+)

Valg $a \in \mathbb{R}_+ : 0 \leq h \leq a \Rightarrow h \cdot f \leq a \cdot f$

$$\stackrel{a \geq 0}{\Rightarrow} (h \cdot f)^+ \leq a \cdot f^+ \wedge (h \cdot f)^- \geq a \cdot f^-$$

$$\stackrel{3.1.4}{\Rightarrow} \int_{\mathcal{X}} (h \cdot f)^+ d\mu \leq \int_{\mathcal{X}} a f^+ d\mu \wedge \int_{\mathcal{X}} (h \cdot f)^- d\mu \leq \int_{\mathcal{X}} a f^- d\mu$$

$$\stackrel{3.1.10}{\Rightarrow} \int_{\mathcal{X}} (h \cdot f)^+ d\mu \leq a \cdot \int_{\mathcal{X}} f^+ d\mu \wedge \int_{\mathcal{X}} (h \cdot f)^- d\mu \leq a \cdot \int_{\mathcal{X}} f^- d\mu$$

$$\stackrel{(*)}{\Rightarrow} \int_{\mathcal{X}} (h \cdot f)^+ d\mu < \infty \vee \int_{\mathcal{X}} (h \cdot f)^- d\mu < \infty, \text{ dvs } h \cdot f \in L(\mu)$$

□

3.4 Sætninger om monoton konvergens

$$(\Omega, \mathcal{F}, \mu) \text{ @ } ; \quad f, (f_n)_{n \geq 1} \in \bar{M}(\Omega, \mathcal{F})$$

$$\text{Påstand: } f_n \uparrow f \text{ } \mu\text{-n.o.} \wedge \int_{\Omega} f_1 d\mu > -\infty \Rightarrow f, (f_n)_{n \geq 1} \in L(\mu) \wedge \int_{\Omega} f d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$$

Bevis:

$$f_n \uparrow f \text{ } \mu\text{-n.o.} \Rightarrow \forall n \geq 1: f_n \geq f_1, f_n \leq f \Rightarrow \forall n \geq 1: 0 \leq f^- \leq f_n^- \leq f_1^- \\ \Rightarrow \forall n \geq 1: \int_{\Omega} f^- d\mu \leq \int_{\Omega} f_n^- d\mu \leq \int_{\Omega} f_1^- d\mu \Rightarrow \forall n \geq 1: f_n \in L(\mu) \wedge f \in L(\mu) \quad \begin{matrix} 3.1.4 & 3.2.2-4 \end{matrix}$$

$$f_n \uparrow f \text{ } \mu\text{-n.o.} \exists N \in \mathcal{F}, \mu(N) = 0 \quad \forall \omega \in \Omega \setminus N: f_n(\omega) \uparrow f(\omega)$$

$$g := 1_{\Omega \setminus N} \cdot f, \quad \forall n \geq 1: g_n := 1_{\Omega \setminus N} \cdot f_n \quad (\text{kbet: } g = f \wedge \forall n \geq 1: g_n = f_n \text{ } \mu\text{-n.o.})$$

$$\text{Bm. } g_n \uparrow g$$

$$g_n \uparrow g \Rightarrow (g_n + g_1^-) \uparrow (g + g_1^-), \quad \forall n \geq 1: 0 \leq g_n + g_1^- \leq g + g_1^- \quad (**)$$

$$\begin{aligned} \int_{\Omega} f d\mu &= \int_{\Omega} g d\mu \stackrel{3.3.4}{=} \int_{\Omega} (g + g_1^-) d\mu - \int_{\Omega} g_1^- d\mu \stackrel{(**) + 3.1.9}{=} \lim_{n \rightarrow \infty} \int_{\Omega} (g_n + g_1^-) d\mu - \int_{\Omega} g_1^- d\mu \\ &= \lim_{n \rightarrow \infty} \left\{ \int_{\Omega} (g_n + g_1^-) d\mu - \int_{\Omega} g_1^- d\mu \right\} \stackrel{3.3.12}{=} \lim_{n \rightarrow \infty} \int_{\Omega} (g_n + g_1^- - g_1^-) d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} g_n d\mu \\ &\stackrel{3.3.4}{=} \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu \quad (*) \end{aligned}$$

□

$$\text{Påstand: } f_n \downarrow f \text{ } \mu\text{-n.o.} \wedge \int_{\Omega} f_1 d\mu < \infty \Rightarrow f, (f_n)_{n \geq 1} \in L(\mu) \wedge \int_{\Omega} f d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$$

Bevis:

$$\left. \begin{aligned} f_n \downarrow f \text{ } \mu\text{-n.o.} &\Rightarrow (-f_n) \uparrow (-f) \text{ } \mu\text{-n.o.} \\ \int_{\Omega} f_1 d\mu < \infty &\stackrel{3.3.11}{\Rightarrow} \int_{\Omega} (-f) d\mu > -\infty \end{aligned} \right\} \stackrel{(+)}{\Rightarrow} \int_{\Omega} (-f) d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} (-f_n) d\mu$$

Af 3.3.11 og 3.2.2-4 følger det ønskede

□

3.5 Fatou's Lemma

$$(\Omega, \mathcal{F}, \mu) \text{ @ } \mu; (f_n)_{n \geq 1} \in \overline{M}(\Omega, \mathcal{F})$$

Påstand:

$$\exists h \in L^1(\mu) \forall n \geq 1: f_n \geq h \text{ } \mu\text{-n.o.} \Rightarrow \int_{\Omega} \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$$

(Bm. specialtilfældet $h \equiv 0$)

Bevis:

$$\exists h \in L^1(\mu) \forall n \geq 1: f_n \geq h \text{ } \mu\text{-n.o.}$$

$$\forall n \geq 1, w \in \Omega: g_n(w) := \inf_{j \geq n} f_j(w) \quad (*)$$

$\Downarrow$

$$\forall n \geq 1: g_n \geq h \text{ } \mu\text{-n.o.}, g_n \leq g_{n+1} \quad (+)$$

$\Downarrow$

$$\liminf_{n \rightarrow \infty} f_n := \sup_{n \geq 1} \left(\inf_{j \geq n} f_j \right) = \sup_{n \geq 1} g_n = \lim_{n \rightarrow \infty} g_n, \text{ dvs. } g_n \uparrow \liminf_{n \rightarrow \infty} f_n$$

$$h \in L^1(\mu) \Rightarrow \int_{\Omega} h d\mu > -\infty \quad \left. \begin{array}{l} \forall n \geq 1: g_n \geq h \text{ } \mu\text{-n.o.} \quad (+) \\ \text{3.3.1} \\ \text{3.3.3} \end{array} \right\} \Rightarrow \forall n \geq 1: g_n \in L^1(\mu) \wedge \int_{\Omega} g_n d\mu > -\infty$$

$$\text{Dvs. } g_n \uparrow \liminf_{n \rightarrow \infty} f_n \wedge \int_{\Omega} g_n d\mu > -\infty \xRightarrow{3.4} \lim_{n \rightarrow \infty} \int_{\Omega} g_n d\mu = \int_{\Omega} \liminf_{n \rightarrow \infty} f_n d\mu \quad (1)$$

$$(*) \Rightarrow \forall n \geq 1, j \geq n: g_n \leq f_j \xRightarrow{\substack{3.3.3 \\ 3.3.1}} \forall n \geq 1, j \geq n: \int_{\Omega} g_n d\mu \leq \int_{\Omega} f_j d\mu$$

$$\Rightarrow \forall n \geq 1: \int_{\Omega} g_n d\mu \leq \inf_{j \geq n} \int_{\Omega} f_j d\mu \quad (4)$$

$$(+)\xRightarrow{3.3.3} \forall n \geq 1: \int_{\Omega} g_n d\mu \leq \int_{\Omega} g_{n+1} d\mu \Rightarrow \sup_{n \geq 1} \int_{\Omega} g_n d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} g_n d\mu$$

$$\stackrel{(4)}{\Rightarrow} \lim_{n \rightarrow \infty} \int_{\Omega} g_n d\mu = \sup_{n \geq 1} \int_{\Omega} g_n d\mu \leq \sup_{n \geq 1} \left(\inf_{j \geq n} \int_{\Omega} f_j d\mu \right) = \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu \quad (2)$$

$$(1) \wedge (2) \Rightarrow \int_{\Omega} \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$$

□

Påstand:

$$\exists h \in L^1(\mu) \forall n \geq 1: f_n \leq h \text{ } \mu\text{-n.o.} \Rightarrow \limsup_{n \rightarrow \infty} \int_{\Omega} f_n d\mu \leq \int_{\Omega} \limsup_{n \rightarrow \infty} f_n d\mu$$

(Bm. specialtilfældet $h \equiv 0$)

Bevis:

$$\exists h \in L^1(\mu) \forall n \geq 1: f_n \leq h \text{ } \mu\text{-n.o.}$$

$$\forall n \geq 1, \omega \in \Omega: g_n(\omega) = \sup_{j \geq n} f_j(\omega) \quad (*)$$

$\Downarrow$

$$\forall n \geq 1: g_n \leq h \text{ } \mu\text{-n.o.}, \quad g_{n+1} \leq g_n \quad (t)$$

$\Downarrow$

$$\limsup_{n \rightarrow \infty} f_n := \inf_{n \geq 1} \left(\sup_{j \geq n} f_j \right) = \inf_{n \geq 1} g_n = \lim_{n \rightarrow \infty} g_n, \text{ dvs. } g_n \downarrow \limsup_{n \rightarrow \infty} f_n$$

$$\left. \begin{array}{l} h \in L^1(\mu) \Rightarrow \int_{\Omega} h d\mu < \infty \\ \forall n \geq 1: g_n \leq h \text{ } \mu\text{-n.o.} \quad (t) \end{array} \right\} \xrightarrow[3.3.3]{3.3.2} \forall n \geq 1: g_n \in L(\mu) \wedge \int_{\Omega} g_1 d\mu < \infty$$

$$\text{Dvs. } g_n \downarrow \limsup_{n \rightarrow \infty} f_n \wedge \int_{\Omega} g_1 d\mu < \infty \xrightarrow[3.4]{3.3.3} \lim_{n \rightarrow \infty} \int_{\Omega} g_n d\mu = \int_{\Omega} \limsup_{n \rightarrow \infty} f_n d\mu \quad (1)$$

$$(*) \Rightarrow \forall n \geq 1, j \geq n: g_n \geq f_j \xrightarrow[3.3.3]{3.3.2} \forall n \geq 1, j \geq n: \int_{\Omega} g_n d\mu \geq \int_{\Omega} f_j d\mu$$

$$\Rightarrow \forall n \geq 1: \int_{\Omega} g_n d\mu \geq \sup_{j \geq n} \int_{\Omega} f_j d\mu \quad (**)$$

$$(t) \xrightarrow[3.3.3]{3.3.2} \forall n \geq 1: \int_{\Omega} g_n d\mu \geq \int_{\Omega} g_{n+1} d\mu \Rightarrow \inf_{n \geq 1} \int_{\Omega} g_n d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} g_n d\mu$$

$$\xRightarrow{(**)} \lim_{n \rightarrow \infty} \int_{\Omega} g_n d\mu = \inf_{n \geq 1} \int_{\Omega} g_n d\mu \geq \inf_{n \geq 1} \left(\sup_{j \geq n} \int_{\Omega} f_j d\mu \right) =: \limsup_{n \rightarrow \infty} \int_{\Omega} f_n d\mu \quad (2)$$

$$(1) \wedge (2) \Rightarrow \int_{\Omega} \limsup_{n \rightarrow \infty} f_n d\mu \geq \limsup_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$$

□

3.6 Lebesgue's sætning om domineret konvergens

$(\mathcal{R}, \mathcal{F}, \mu) \text{ @}$; $f, (f_n)_{n \geq 1} \in \mathcal{F}(\mathcal{R}, \mathcal{F})$; $f_n \rightarrow f$ μ -n.o.

Påstand:

$\exists h \in L^1(\mu) \forall n \geq 1: |f_n| \leq h$ μ -n.o. , da gælder, at $f, (f_n)_{n \geq 1} \in L^1(\mu)$ og

$$\int_{\mathcal{R}} f d\mu = \lim_{n \rightarrow \infty} \int_{\mathcal{R}} f_n d\mu \quad (3.6.1)$$

$$\lim_{n \rightarrow \infty} \int_{\mathcal{R}} |f - f_n| d\mu = 0 \quad (3.6.2)$$

Bevis:

$\exists h \in L^1(\mu) \forall n \geq 1: |f_n| \leq h \wedge f_n \rightarrow f$ μ -n.o. $\Rightarrow |f| \leq h$ μ -n.o.

$\xRightarrow[3.3.5]{} f, (f_n)_{n \geq 1} \in L^1(\mu)$

$\forall n \geq 1: |f_n| \leq h$ μ -n.o. $\Rightarrow \forall n \geq 1: -h \leq f_n \leq h$ μ -n.o.

$\xRightarrow[h \in L^1(\mu), 3.5]{} \limsup_{n \rightarrow \infty} \int_{\mathcal{R}} f_n d\mu \leq \int_{\mathcal{R}} \limsup_{n \rightarrow \infty} f_n d\mu \wedge \int_{\mathcal{R}} \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_{\mathcal{R}} f_n d\mu \quad (*)$

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\mathcal{R}} f_n d\mu &\leq \int_{\mathcal{R}} \limsup_{n \rightarrow \infty} f_n d\mu \stackrel{\substack{\text{P.4.22} \\ 3.3.4}}{=} \int_{\mathcal{R}} f d\mu \stackrel{\substack{\text{P.4.22} \\ 3.3.4}}{=} \int_{\mathcal{R}} \liminf_{n \rightarrow \infty} f_n d\mu \\ &\stackrel{(*)}{\leq} \liminf_{n \rightarrow \infty} \int_{\mathcal{R}} f_n d\mu \end{aligned}$$

Trivielt gælder, at $\liminf_{n \rightarrow \infty} \int_{\mathcal{R}} f_n d\mu \leq \limsup_{n \rightarrow \infty} \int_{\mathcal{R}} f_n d\mu$

Alt i alt kan vi konkludere, at $\liminf_{n \rightarrow \infty} \int_{\mathcal{R}} f_n d\mu = \int_{\mathcal{R}} f d\mu = \limsup_{n \rightarrow \infty} \int_{\mathcal{R}} f_n d\mu$

Af P.4.22 kan vi konkludere, at $\lim_{n \rightarrow \infty} \int_{\mathcal{R}} f_n d\mu = \int_{\mathcal{R}} f d\mu$

Idet, $f_n \xrightarrow{\mu\text{-n.o.}} f \Rightarrow |f - f_n| \xrightarrow{\mu\text{-n.o.}} 0$ og

$\forall n \geq 1: |f_n| \leq h \wedge |f| \leq h$ μ -n.o. $\Rightarrow \forall n \geq 1: |f - f_n| \leq 2 \cdot h$ μ -n.o.,

samt $2 \cdot h \in L^1(\mu)$ jvf. 3.3.10, følger det ønskede af det netop viste resultat $\square$

3.7 Beppo Levi's sætning

$(\mathcal{R}, \mathcal{F}, \mu) @, (f_n)_{n \geq 1} \in \bar{M}(\mathcal{R}, \mathcal{F})$

Påstand:

Hvis $\sum_{n=1}^{\infty} \int_{\mathcal{R}} f_n^+ d\mu < \infty$, da gælder

$$\sum_{n=1}^{\infty} f_n \text{ eksisterer } < \infty \quad \mu\text{-n.o.} \quad (3.7.1)$$

$$\int_{\mathcal{R}} \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_{\mathcal{R}} f_n d\mu \quad (3.7.2)$$

Beweis:

$$g := \sum_{n=1}^{\infty} f_n^+, \quad h := \sum_{n=1}^{\infty} f_n^- \quad (g, h \text{ er veldef. idet } \forall n \geq 1: f_n^+, f_n^- \geq 0)$$

$$\sum_{k=1}^n f_k^+ \uparrow g \wedge \int_{\mathcal{R}} f_1^+ \geq 0 > -\infty \xRightarrow{3.4} \int_{\mathcal{R}} g d\mu = \lim_{n \rightarrow \infty} \int_{\mathcal{R}} \sum_{k=1}^n f_k^+ d\mu = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_{\mathcal{R}} f_k^+ d\mu = \sum_{k=1}^{\infty} \int_{\mathcal{R}} f_k^+ d\mu$$

$$\sum_{k=1}^n f_k^- \uparrow h \wedge \int_{\mathcal{R}} f_1^- \geq 0 > -\infty \xRightarrow{3.4} \int_{\mathcal{R}} h d\mu = \lim_{n \rightarrow \infty} \int_{\mathcal{R}} \sum_{k=1}^n f_k^- d\mu = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_{\mathcal{R}} f_k^- d\mu = \sum_{k=1}^{\infty} \int_{\mathcal{R}} f_k^- d\mu \quad (*)$$

Det antages, at $\sum_{n=1}^{\infty} \int_{\mathcal{R}} f_n^+ d\mu < \infty$ følger af ovenstående, at $\int_{\mathcal{R}} g d\mu < \infty$. (*)

Dermed følger af 3.3.6, at $g < \infty$ μ -n.o. (+)

$$(+)\Rightarrow \infty > g - h = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n f_k^+ \right) - \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n f_k^- \right) = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n (f_k^+ - f_k^-) \right) = \sum_{n=1}^{\infty} f_n \quad \mu\text{-n.o.}$$

Idet $\sum_{n=1}^{\infty} f_n \leq g$ μ -n.o. og $\int_{\mathcal{R}} g d\mu < \infty$ (*) følger af 3.3.2,

at $\sum_{n=1}^{\infty} f_n \in L(\mu)$

$$\begin{aligned} \int_{\mathcal{R}} \sum_{n=1}^{\infty} f_n d\mu &= \int_{\mathcal{R}} (g - h) d\mu \stackrel{3.3.11-12}{=} \int_{\mathcal{R}} g d\mu - \int_{\mathcal{R}} h d\mu \stackrel{(*)}{=} \sum_{k=1}^{\infty} \int_{\mathcal{R}} f_k^+ d\mu - \sum_{k=1}^{\infty} \int_{\mathcal{R}} f_k^- d\mu \\ &= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \int_{\mathcal{R}} f_k^+ d\mu - \int_{\mathcal{R}} f_k^- d\mu \right) \stackrel{3.3.11-12}{=} \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \int_{\mathcal{R}} f_k^+ - f_k^- d\mu \right) = \sum_{n=1}^{\infty} \int_{\mathcal{R}} f_n d\mu \end{aligned}$$

□

Bmk. at 3.7.2 holder når $\forall n \geq 1: f_n \leq 0$ μ -n.o.

Påstand:

Hvis $\sum_{n=1}^{\infty} \int_{\mathcal{X}} f_n^- d\mu < \infty$, da gælder

$$\sum_{n=1}^{\infty} f_n \text{ eksisterer } > -\infty \quad \mu\text{-n.o.} \quad (3.7.3)$$

$$\int_{\mathcal{X}} \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_{\mathcal{X}} f_n d\mu \quad (3.7.4)$$

Bevis:

$$g := \sum_{n=1}^{\infty} f_n^+, \quad h := \sum_{n=1}^{\infty} f_n^- \quad (g, h \text{ er veldef. idet } \forall n \geq 1: f_n^+, f_n^- \geq 0)$$

$$\begin{aligned} \sum_{k=1}^n f_k^+ \uparrow g \wedge \int_{\mathcal{X}} f_1^+ d\mu \geq 0 > -\infty &\xRightarrow{3.1.4} \int_{\mathcal{X}} g d\mu = \lim_{n \rightarrow \infty} \int_{\mathcal{X}} \sum_{k=1}^n f_k^+ d\mu = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_{\mathcal{X}} f_k^+ d\mu = \sum_{k=1}^{\infty} \int_{\mathcal{X}} f_k^+ d\mu \\ \sum_{k=1}^n f_k^- \uparrow h \wedge \int_{\mathcal{X}} f_1^- d\mu \geq 0 > -\infty &\xRightarrow{3.1.4} \int_{\mathcal{X}} h d\mu = \lim_{n \rightarrow \infty} \int_{\mathcal{X}} \sum_{k=1}^n f_k^- d\mu = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_{\mathcal{X}} f_k^- d\mu = \sum_{k=1}^{\infty} \int_{\mathcal{X}} f_k^- d\mu \end{aligned} \quad (*)$$

$$\text{Vi antager, at } \sum_{n=1}^{\infty} \int_{\mathcal{X}} f_n^- d\mu < \infty \xRightarrow{(*)} \int_{\mathcal{X}} h d\mu < \infty \xRightarrow{3.3.6} h < \infty \quad \mu\text{-n.o.}$$

$$\Rightarrow -\infty < g - h = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n f_k^+ \right) - \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n f_k^- \right) = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n (f_k^+ - f_k^-) \right) = \sum_{n=1}^{\infty} f_n \quad \mu\text{-n.o.}$$

$$\text{Idet } \sum_{n=1}^{\infty} f_n \geq -h \quad \mu\text{-n.o.} \wedge \int_{\mathcal{X}} (-h) d\mu = -\int_{\mathcal{X}} h d\mu > -\infty \text{ gælder jef. 3.3.1 } \sum_{n=1}^{\infty} f_n \text{ eks.}$$

$$\begin{aligned} \int_{\mathcal{X}} \sum_{n=1}^{\infty} f_n d\mu &= \int_{\mathcal{X}} (g - h) d\mu = \int_{\mathcal{X}} g d\mu - \int_{\mathcal{X}} h d\mu = \sum_{k=1}^{\infty} \int_{\mathcal{X}} f_k^+ d\mu - \sum_{k=1}^{\infty} \int_{\mathcal{X}} f_k^- d\mu \\ &= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \int_{\mathcal{X}} f_k^+ d\mu - \sum_{k=1}^n \int_{\mathcal{X}} f_k^- d\mu \right) = \lim_{n \rightarrow \infty} \int_{\mathcal{X}} \left(\sum_{k=1}^n (f_k^+ - f_k^-) \right) d\mu = \sum_{n=1}^{\infty} \int_{\mathcal{X}} f_n d\mu \end{aligned}$$

□

Bmk. at 3.7.4 holder når $\forall n \geq 1: f_n \geq 0 \quad \mu\text{-n.o.}$

3.8 Integraler af simple og diskrete funktioner

$$(\mathcal{X}, \mathcal{F}, \mu) @ \mathbb{W}, f \in \mathcal{F}$$

$$f := 1_F$$

$$j, n \in \mathbb{N}: \{f > j \cdot 2^{-n}\} = \begin{cases} F, & 1 \leq j \leq 2^n - 1 \\ \emptyset, & j \geq 2^n \end{cases}$$

$$s_n(f) = 2^{-n} \cdot \sum_{j=1}^{\infty} \mu(f > j \cdot 2^{-n}) = 2^{-n} \cdot (2^n - 1) \mu(F) = (1 - 2^{-n}) \mu(F)$$

$$\int_F 1 d\mu := \int_{\mathcal{X}} 1_F d\mu \stackrel{3.2.8}{=} \lim_{n \rightarrow \infty} s_n(f) = \lim_{n \rightarrow \infty} \{(1 - 2^{-n}) \mu(F)\} = \mu(F) \quad (3.8.1)$$

Påstand:

Hvis $(a_n)_{n \geq 1} \in \mathbb{R}$, $(F_n)_{n \geq 1} \in \mathcal{F}$: $\sum_{n=1}^{\infty} a_n^+ \mu(F_n) < \infty \vee \sum_{n=1}^{\infty} a_n^- \mu(F_n) < \infty$, da gælder

$$\int_{\mathcal{X}} \left(\sum_{n=1}^{\infty} a_n 1_{F_n} \right) d\mu = \sum_{n=1}^{\infty} a_n \mu(F_n)$$

Specielt gælder, at hvis $f \in \bar{M}(\mathcal{X}, \mathcal{F}) \exists (a_n)_{n \geq 1} \in \mathbb{R}; (F_n)_{n \geq 1} \in \mathcal{F} \forall w \in \mathcal{X}: f(w) = \sum_{n=1}^{\infty} a_n \cdot 1_{F_n}(w)$
og $\sum_{n=1}^{\infty} a_n^+ \cdot \mu(f = a_n) < \infty \vee \sum_{n=1}^{\infty} a_n^- \cdot \mu(f = a_n) < \infty$: $\int_{\mathcal{X}} f d\mu = \sum_{n=1}^{\infty} a_n \cdot \mu(f = a_n)$

Beweis:

$$(a_n)_{n \geq 1} \in \mathbb{R}, (F_n)_{n \geq 1} \in \mathcal{F}$$

$$\text{Antag } \sum_{n=1}^{\infty} a_n^+ \mu(F_n) < \infty$$

$$\sum_{n=1}^{\infty} \int_{\mathcal{X}} (a_n 1_{F_n})^+ d\mu = \sum_{n=1}^{\infty} \int_{\mathcal{X}} a_n^+ 1_{F_n} d\mu \stackrel{3.3.11}{=} \sum_{n=1}^{\infty} a_n^+ \int_{\mathcal{X}} 1_{F_n} d\mu \stackrel{3.2.1}{=} \sum_{n=1}^{\infty} a_n^+ \mu(F_n) < \infty$$

$\Downarrow$ 3.7 (Beppo Levi's sætning)

$$\int_{\mathcal{X}} \left(\sum_{n=1}^{\infty} a_n 1_{F_n} \right) d\mu = \sum_{n=1}^{\infty} \int_{\mathcal{X}} a_n 1_{F_n} d\mu \stackrel{3.3.11}{=} \sum_{n=1}^{\infty} a_n \int_{\mathcal{X}} 1_{F_n} d\mu \stackrel{3.2.1}{=} \sum_{n=1}^{\infty} a_n \mu(F_n)$$

Tilsvarende bevis når $\sum_{n=1}^{\infty} a_n^- \mu(F_n) < \infty$

Specialtilfældet følger af lighedene $f = \sum_{n=1}^{\infty} a_n 1_{F_n} \Rightarrow \mu(F_n) = \mu(f = a_n)$

□

3.9 Markov's ulighed

$(\mathcal{X}, \mathcal{F}, \mu) \text{ @}$, $\varphi: \bar{\mathbb{R}}_+ \rightarrow \bar{\mathbb{R}}_+$, $\forall x, y \in \bar{\mathbb{R}}_+ : x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$

Præstand:

$$a > 0: 0 < \varphi(a) < \infty \Rightarrow \mu(|f| > a) \leq \frac{1}{\varphi(a)} \int_{\mathcal{X}} \varphi(|f|) d\mu \quad (3.9.1)$$

Bewis:

$$a > 0: 0 < \varphi(a) < \infty$$

$$F := \{|f| > a\} ; \quad g: F \rightarrow \bar{\mathbb{R}}_+ \quad \forall w \in F: g(w) := \varphi(|f(w)|)$$

$$\forall w \in F: g(w) \geq \varphi(a) > 0, \text{ dvs. } g \geq \varphi(a) \cdot 1_F \geq 0$$

$\Downarrow$

$$\int_{\mathcal{X}} g d\mu \underset{3.1.4}{\geq} \int_{\mathcal{X}} \varphi(a) \cdot 1_F d\mu \underset{3.1.10}{=} \varphi(a) \cdot \int_{\mathcal{X}} 1_F d\mu \underset{3.8.1}{=} \varphi(a) \cdot \mu(F)$$

$\Downarrow$ (idet $\varphi(a) > 0$)

$$\frac{1}{\varphi(a)} \cdot \int_{\mathcal{X}} g d\mu \geq \mu(F), \text{ dvs. } \mu(|f| > a) \leq \frac{1}{\varphi(a)} \cdot \int_{\mathcal{X}} \varphi(|f|) d\mu \text{ pr. def. af } F \text{ og } g \quad \square$$

3.10 Jensen's ulighed

$(\Omega, \mathcal{F}, \mu) \text{ (m)}$

$$F \in \mathcal{F} : 0 < \mu(F) < \infty \quad (F \in \mathcal{F} \setminus \emptyset / \mu \wedge \mu(F) < \infty)$$

$$f \in L^1(\mu), \exists a, b \in \mathbb{R} : f(\omega) \in [a, b] =: I$$

$$\varphi : I \rightarrow \mathbb{R}, \forall s, t \in I, 0 \leq \alpha \leq 1 : \varphi(\alpha s + (1-\alpha) \cdot t) \leq \alpha \cdot \varphi(s) + (1-\alpha) \cdot \varphi(t) \quad (\text{dvs. } \varphi \text{ konvex})$$

Påstand:

$$\frac{1}{\mu(F)} \int_F f d\mu \in I \quad (3.10.1)$$

$$\varphi\left(\frac{1}{\mu(F)} \int_F f d\mu\right) \leq \frac{1}{\mu(F)} \int_F \varphi(f) d\mu \quad (3.10.2)$$

Bevis:

$$m := \frac{1}{\mu(F)} \int_F f d\mu$$

$$f(\omega) \in I \Rightarrow a \cdot 1_F \leq f \cdot 1_F \leq b \cdot 1_F \quad (*)$$

Af 3.8.1 følger, at $\int_{\Omega} a \cdot 1_F d\mu = a \cdot \mu(F) < \infty$, $\int_{\Omega} b \cdot 1_F d\mu = b \cdot \mu(F) < \infty$,
dvs. at $f \cdot 1_F \in L(\mu)$ jvf (*) og 3.3.2.

$$3.3.3 \text{ og } (*) \text{ giver nu, at } a \cdot \mu(F) = \int_{\Omega} a \cdot 1_F d\mu \leq \int_{\Omega} f \cdot 1_F d\mu \leq \int_{\Omega} b \cdot 1_F d\mu = b \cdot \mu(F)$$

$\Downarrow$ (da $\mu(F) > 0$)

$$a \leq \frac{1}{\mu(F)} \int_{\Omega} f \cdot 1_F d\mu = \frac{1}{\mu(F)} \int_F f d\mu =: m \leq b, \text{ altså } a \leq m \leq b$$

Antag, at $m=a \in I$, da gælder:

$$\begin{aligned} \int_{\Omega} 1_F (f-a) d\mu &\stackrel{3.3.12}{=} \int_{\Omega} 1_F \cdot f d\mu - \int_{\Omega} 1_F \cdot a d\mu \stackrel{3.3.9}{=} \int_{\Omega} 1_F \cdot f d\mu - a \cdot \int_{\Omega} 1_F d\mu \\ &\stackrel{\substack{3.2.8 \\ 3.8.1}}{=} \int_F f d\mu - a \cdot \mu(F) \stackrel{m=a}{=} \int_F f d\mu - \int_F f d\mu = 0 \end{aligned}$$

$$1_F (f-a) \geq 0 \text{ idet } f(F) \subseteq I = [a, b]$$

$\Downarrow$

$$\int_{\Omega} 1_F (f-a) d\mu = \int_{\Omega} |1_F (f-a)| d\mu = 0$$

$\Downarrow$ 3.3.8

$$1_F (f-a) = 0 \text{ } \mu\text{-n.o.}$$

$\Downarrow$

$$f=a \text{ } \mu\text{-n.o. på } F \quad (+)$$

$$\begin{aligned} \varphi(m) &\stackrel{3.8.1}{=} \varphi(m) \cdot \frac{1}{\mu(F)} \int_F d\mu \stackrel{\substack{3.2.8 \\ 3.3.11}}{=} \frac{1}{\mu(F)} \cdot \int_F \varphi(m) d\mu \stackrel{m=a}{=} \frac{1}{\mu(F)} \cdot \int_F \varphi(a) d\mu \\ &\stackrel{(+) \substack{3.3.4}}{=} \frac{1}{\mu(F)} \cdot \int_F \varphi(f) d\mu \end{aligned} \quad (*)$$

For $m=b$ fås $1_F (b-f) \geq 0 \text{ } \mu\text{-n.o.}$ og dermed $f=b \text{ } \mu\text{-n.o. på } F$.

Som i (*) følger, at $\varphi(m) = \frac{1}{\mu(F)} \int_F \varphi(f) d\mu$

Vi har tidligere vist 3.10.1: $m \in I$

Sætning 3.10.2 er vist i tilfældene $m=a$ og $m=b$

Antag, at $m \in]a, b[$

Idet φ er konveks, eksisterer φ 's højreafledede på $]a, b[$ (Exc. 3.33)

λ betegner φ 's højreafledede i pkt. et m

Idet φ er konveks gælder: $\forall x \in I: \varphi(x) \geq \lambda(x-m) + \varphi(m)$ (*)

$$(*) \wedge f(F) \in I \Rightarrow \forall \omega \in F: \varphi(f(\omega)) \geq \lambda(f(\omega)-m) + \varphi(m)$$

$$\Rightarrow \forall \omega \in \Omega: \varphi(f(\omega)) 1_F(\omega) \geq (\lambda(f(\omega)-m) + \varphi(m)) 1_F(\omega)$$

$$\stackrel{3.3.3}{\Rightarrow} \stackrel{3.2.8}{\int_F} \varphi(f) d\mu \geq \int_F (\lambda(f-m) + \varphi(m)) d\mu$$

$$\stackrel{3.3.11-12}{=} \stackrel{3.8.1}{\lambda \cdot \int_F f d\mu - \lambda \cdot m \cdot \mu(F) + \varphi(m) \cdot \mu(F)}$$

$$= \varphi\left(\frac{1}{\mu(F)} \int_F f d\mu\right) \cdot \mu(F)$$

$$\stackrel{\mu(F) > 0}{\Rightarrow} \varphi\left(\frac{1}{\mu(F)} \int_F f d\mu\right) \leq \frac{1}{\mu(F)} \cdot \int_F \varphi(f) d\mu$$

□

3.11 Hölders Ulighed

$$(\Omega, \mathcal{F}, \mu) @; f, g \in \bar{M}(\Omega, \mathcal{F}); r, p, q \in \mathbb{R}_+ : \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \quad (*)$$

Påstand:

$$\left(\int_{\Omega} |fg|^r d\mu \right)^{1/r} \leq \left(\int_{\Omega} |f|^p d\mu \right)^{1/p} \left(\int_{\Omega} |g|^q d\mu \right)^{1/q} \quad (3.11.1)$$

(for $r=1, p=q=2$ benævnes denne ulighed Cauchy-Schwarz' ulighed)

Bevis:

$$\alpha := \left(\int_{\Omega} |f|^p d\mu \right)^{1/p}, \quad \beta := \left(\int_{\Omega} |g|^q d\mu \right)^{1/q}, \quad \alpha, \beta \geq 0 \text{ jvf. 3.1.4}$$

$$\alpha = 0 \Rightarrow \int_{\Omega} |f|^p d\mu = 0 \xrightarrow{3.3.8} f = 0 \text{ } \mu\text{-n.o.} \Rightarrow |fg|^r = 0 \text{ } \mu\text{-n.o.} \xrightarrow{3.3.8} \int_{\Omega} |fg|^r d\mu = 0$$

$$\beta = 0 \Rightarrow \int_{\Omega} |g|^q d\mu = 0 \Rightarrow g = 0 \text{ } \mu\text{-n.o.} \Rightarrow |fg|^r = 0 \text{ } \mu\text{-n.o.} \xrightarrow{3.3.8} \int_{\Omega} |fg|^r d\mu = 0$$

Vi ser, at for $\alpha=0$ el. $\beta=0$ er 3.11.1 opfyldt.

$\alpha=\infty \wedge \beta \neq 0$ el. $\alpha \neq 0 \wedge \beta=\infty$ er 3.11.1 trivielt opfyldt, idet $\left(\int_{\Omega} |fg|^r d\mu \right)^{1/r} \in [0, \infty]$

Antag i det følgende, at $0 < \alpha < \infty \wedge 0 < \beta < \infty$

Den naturlige logaritme er konkav, dvs.

$$\forall x, y > 0, 0 \leq t \leq 1: \log(t \cdot x + (1-t) \cdot y) \geq t \cdot \log(x) + (1-t) \cdot \log(y)$$

$$\Rightarrow t \cdot x + (1-t) \cdot y = \exp(\log(t \cdot x + (1-t) \cdot y)) \geq \exp(t \cdot \log(x) + (1-t) \cdot \log(y)) \\ = x^t \cdot y^{(1-t)}$$

Idet $t \cdot 0 + (1-t) \cdot y = (1-t) \cdot y \geq 0 = 0^t \cdot y^{(1-t)}$, $t \cdot x + (1-t) \cdot 0 = t \cdot x \geq 0 = x^t \cdot 0^{(1-t)}$
for $0 < t < 1$ fås: $\forall x, y \geq 0; 0 < t < 1: t \cdot x + (1-t) \cdot y \geq x^t \cdot y^{(1-t)} \quad (*)$

$$t := \frac{r}{p}, \quad 0 < \frac{r}{p} \stackrel{(*)}{=} \frac{(1/p + 1/q)^{-1}}{p} = \frac{1}{1 + p/q} < 1, \quad \text{dus. } 0 < t < 1$$

$$1-t = 1 - \frac{r}{p} \stackrel{(*)}{=} 1 - r \cdot \left(\frac{1}{r} - \frac{1}{q} \right) = \frac{r}{q}$$

$$x := |1/\alpha \cdot f|^p, \quad y := |1/\beta \cdot g|^q \quad (\text{husk } 0 < \alpha, \beta < \infty)$$

Detta indsättes i (*):

$$\begin{aligned} t \cdot x + (1-t) \cdot y &= t \cdot |1/\alpha \cdot f|^p + (1-t) \cdot |1/\beta \cdot g|^q = t \cdot \frac{1}{\alpha^p} |f|^p + (1-t) \cdot \frac{1}{\beta^q} |g|^q \\ &\geq x^t \cdot y^{(1-t)} = (|1/\alpha \cdot f|^p)^{t/p} \cdot (|1/\beta \cdot g|^q)^{(1-t)/q} = \alpha^{-r} \beta^{-r} \cdot |f \cdot g|^r \end{aligned}$$

$$\Downarrow \text{ 3.3.3 + 3.3.11-12}$$

$$\int_{\mathbb{R}} |f \cdot g|^r d\mu \leq (\alpha \cdot \beta)^r \cdot \left(t \cdot \frac{1}{\alpha^p} \int_{\mathbb{R}} |f|^p d\mu + (1-t) \cdot \frac{1}{\beta^q} \int_{\mathbb{R}} |g|^q d\mu \right) = (\alpha \beta)^r (t + 1-t)$$

def. på α, β

$\Downarrow$

$$\left(\int_{\mathbb{R}} |f g|^r d\mu \right)^{1/r} \leq \alpha \cdot \beta$$

□

3.12 Minkowski's Uighed

$(\Omega, \mathcal{F}, \mu) @ \mu$; $f, g \in \bar{F}(\Omega, \mathcal{F})$

Påstand:

$$1 \leq p < \infty : \left(\int_{\Omega} |f+g|^p d\mu \right)^{1/p} \leq \left(\int_{\Omega} |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} |g|^p d\mu \right)^{1/p} \quad (3.12.1)$$

$\varphi: \mathbb{R}_{+,0} \rightarrow \mathbb{R}_{+,0}$ voksende og konkav:

$$\int_{\Omega} \varphi(|f+g|) d\mu \leq \int_{\Omega} \varphi(|f|) d\mu + \int_{\Omega} \varphi(|g|) d\mu \quad (3.12.2)$$

(specielt følger af 3.12.2, at $0 < p \leq 1$: $\int_{\Omega} |f+g|^p d\mu \leq \int_{\Omega} |f|^p d\mu + \int_{\Omega} |g|^p d\mu$ (3.12.3))

Bevis:

3.12.2:

Lad $x, y \geq 0$ være vilkårlige. $\lambda = \frac{x}{x+y}$ ($\lambda := 0$ for $x=0 \wedge y=0$) Bm. $\lambda \in [0,1]$

Vi ser, at $x = \lambda(x+y) + (1-\lambda) \cdot 0$, $y = (1-\lambda)(x+y) + \lambda \cdot 0$

φ er konkav, dvs.

$$\varphi(x) \geq \lambda \cdot \varphi(x+y) + (1-\lambda) \cdot \varphi(0) \geq \lambda \cdot \varphi(x+y) \quad (1-\lambda) \in [0,1], \varphi(0) \geq 0$$

$$\varphi(y) \geq (1-\lambda) \cdot \varphi(x+y) + \lambda \cdot \varphi(0) \geq (1-\lambda) \cdot \varphi(x+y) \quad \lambda \in [0,1], \varphi(0) \geq 0$$

$\Downarrow$

$$\lambda \cdot \varphi(x+y) + (1-\lambda) \cdot \varphi(x+y) = \varphi(x+y) \leq \varphi(x) + \varphi(y)$$

Idet $0 \leq |f+g| \leq |f|+|g|$ og φ er voksende får (sæt $x=|f|$, $y=|g|$):

$$\varphi(|f+g|) \leq \varphi(|f|+|g|) \leq \varphi(|f|) + \varphi(|g|)$$

$\Downarrow$ 3.3.3 + 3.3.12

$$\int_{\Omega} \varphi(|f+g|) d\mu \leq \int_{\Omega} \varphi(|f|) d\mu + \int_{\Omega} \varphi(|g|) d\mu$$

3.12.3: Benyt 3.12.2 med $\phi(x) = x^p$

3.12.1:

For $p=1$ følger påstanden af 3.12.3, antag derfor, at $1 < p < \infty$

Sat $r=1$, $p=p$, $q = \frac{p}{p-1}$ og benyt Hölders Ulighed:

$$\int_{\Omega} |f| |f+g|^{p-1} d\mu \leq \left\{ \int_{\Omega} |f|^p d\mu \right\}^{1/p} \cdot \left\{ \int_{\Omega} |f+g|^{q \cdot (p-1)} d\mu \right\}^{1/q} \quad (*)$$

$$\int_{\Omega} |g| |f+g|^{p-1} d\mu \leq \left\{ \int_{\Omega} |g|^p d\mu \right\}^{1/p} \cdot \left\{ \int_{\Omega} |f+g|^{q \cdot (p-1)} d\mu \right\}^{1/q}$$

Trekantuligheden giver: $|f+g|^p = |f+g| |f+g|^{p-1} \leq (|f| + |g|) |f+g|^{p-1} \quad (+)$

$$\begin{aligned} \int_{\Omega} |f+g|^p d\mu &\stackrel{\substack{3.3.3 \\ + (+) \\ + 3.3.12}}{\leq} \int_{\Omega} |f| |f+g|^{p-1} d\mu + \int_{\Omega} |g| |f+g|^{p-1} d\mu \\ &\stackrel{(*)}{\leq} \left(\left\{ \int_{\Omega} |f|^p d\mu \right\}^{1/p} + \left\{ \int_{\Omega} |g|^p d\mu \right\}^{1/p} \right) \cdot \left\{ \int_{\Omega} |f+g|^{q \cdot (p-1)} d\mu \right\}^{1/q} \\ &\stackrel{p=q(p-1)}{=} \left(\left\{ \int_{\Omega} |f|^p d\mu \right\}^{1/p} + \left\{ \int_{\Omega} |g|^p d\mu \right\}^{1/p} \right) \cdot \left\{ \int_{\Omega} |f+g|^p d\mu \right\}^{1/q} \end{aligned}$$

$\Downarrow$

$$\left(\int_{\Omega} |f+g|^p d\mu \right)^{1/p} \stackrel{1 - 1/q = 1/p}{=} \left\{ \int_{\Omega} |f+g|^p d\mu \right\}^{-1/q} \cdot \int_{\Omega} |f+g|^p d\mu \leq \left\{ \int_{\Omega} |f|^p d\mu \right\}^{1/p} + \left\{ \int_{\Omega} |g|^p d\mu \right\}^{1/p}$$

□

3.13

 $(\Omega, \mathcal{F}, \mu) @ \quad , \quad f \in L(\mu) \quad , \quad n \geq 1: F_n \in \mathcal{F}, F \in \mathcal{F}$

Påstand:

$$\text{Hvis } (F_n \uparrow F) \vee (F_n \downarrow F \wedge f \in L^1(\mu)): \int_{F_n} f d\mu \xrightarrow{n \rightarrow \infty} \int_F f d\mu \quad (3.13.1)$$

$$\text{Hvis } \{F_n\}_{n \geq 1} \text{ disjunkte } \wedge \bigcup_{n=1}^{\infty} F_n = F: \int_F f d\mu = \sum_{n=1}^{\infty} \int_{F_n} f d\mu \quad (3.13.2)$$

$$\text{Hvis } f \in L^1(\mu): \forall \varepsilon > 0 \exists \delta > 0: \mu(A) \leq \delta \Rightarrow \left| \int_A f d\mu \right| \leq \varepsilon \quad (3.13.3)$$

(i kort notation: $\lim_{\mu(A) \rightarrow 0} \int_A f d\mu = 0$)

Bewis:

3.13.1:

$$\text{Antag } F_n \uparrow F \Rightarrow 1_{F_n} f^+ \uparrow 1_F f^+ \wedge 1_{F_n} f^- \uparrow 1_F f^-$$

$$\int_{\Omega} 1_{F_n} f^+ d\mu, \int_{\Omega} 1_{F_n} f^- d\mu \geq 0 > -\infty$$

Af Sætn. om monoton konvergens (3.4) følger, at

$$\lim_{n \rightarrow \infty} \int_{F_n} f^+ d\mu = \int_F f^+ d\mu \quad \text{og} \quad \lim_{n \rightarrow \infty} \int_{F_n} f^- d\mu = \int_F f^- d\mu$$

Idet $f \in L(\mu): \int_F f^+ d\mu < \infty \vee \int_F f^- d\mu < \infty$ og af vurderingerne

$$\forall n \geq 1: \int_{F_n} f^+ d\mu \leq \int_F f^+ d\mu \wedge \int_{F_n} f^- d\mu \leq \int_F f^- d\mu \quad (3.3.3)$$

$$\text{gælder } \int_F f d\mu = \int_F f^+ d\mu - \int_F f^- d\mu = \lim_{n \rightarrow \infty} \left(\int_{F_n} f^+ d\mu - \int_{F_n} f^- d\mu \right) = \lim_{n \rightarrow \infty} \int_{F_n} f d\mu$$

Antag $F_n \downarrow F$ og $f \in L^1(\mu)$

$f \in L^1(\mu) \Rightarrow 1_{F_n} f \in L^1(\mu)$, $\forall n \geq 1: |1_{F_n} f| \leq |f|$ idet $F_n \downarrow F$

Af Lebesgues sætn. om domineret konvergens følger, at $\int_F f d\mu = \lim_{n \rightarrow \infty} \int_{F_n} f d\mu$

3.13.2:

Antag $\{F_n\}_{n \geq 1}$ disjunkte og $\bigcup_{n=1}^{\infty} F_n = F$

$$1_F f = \sum_{n=1}^{\infty} 1_{F_n} f, \quad 1_F f^+ = \sum_{n=1}^{\infty} 1_{F_n} f^+, \quad 1_F f^- = \sum_{n=1}^{\infty} 1_{F_n} f^- \quad (*)$$

Idet $f \in L(\mu) \Rightarrow 1_F f \in L(\mu) \Rightarrow \int_F f^+ d\mu < \infty \vee \int_F f^- d\mu < \infty$

Af 3.1.8 og (*) følger: $\sum_{n=1}^{\infty} \int_{F_n} f^+ d\mu < \infty \vee \sum_{n=1}^{\infty} \int_{F_n} f^- d\mu < \infty$

Beppo Levi's sætn. (3.7) giver dermed $\int_F f d\mu = \sum_{n=1}^{\infty} \int_{F_n} f d\mu$

3.13.3

Antag $f \in L^1(\mu)$

Lad $\varepsilon > 0$ være givet

Definer $\forall A \in \mathcal{F}: v(A) = \int_A |f| d\mu$

Idet $v(\emptyset) = \int_{\emptyset} |f| d\mu = 0$, $v: \mathcal{F} \rightarrow \mathbb{R}_+^{(*)}$ og v er σ -additiv (jvf. 3.13.2) ser vi, at v er et mål på $\mathcal{F}$.

$$v(\Omega) < \infty \stackrel{(*)}{\iff} \mu(|f| = \infty) = 0 \stackrel{3.3.8}{\implies} v(|f| = \infty) = 0$$

$$v(|f| > 1) \leq v(\Omega) < \infty, \quad \{|f| > n\} \downarrow \{|f| = \infty\}$$

Af 1.4.3 følger $\lim_{n \rightarrow \infty} v(|f| > n) = 0$, dvs. $\exists p \geq 1: v(|f| > p) \leq \varepsilon/2$

(*) : $v(\Omega) < \infty$, da $f \in L^1(\mu)$

$$\delta := \frac{\varepsilon}{2p}$$

$$\text{Betragt } A \in \mathcal{F} : \mu(A) \leq \delta \quad (+)$$

$$v(A \cap \{|f| \leq p\}) = \int_{A \cap \{|f| \leq p\}} |f| d\mu \stackrel{3.10.1}{\leq} p \cdot \mu(A \cap \{|f| \leq p\}) \stackrel{1.3.2}{\leq} p \cdot \mu(A) \quad (*)$$

$$\begin{aligned} \int_A |f| d\mu = v(A) &= v(A \cap \{|f| > p\}) + v(A \cap \{|f| \leq p\}) \stackrel{(*)}{\leq} v(|f| > p) + p \cdot \mu(A) \\ &\stackrel{(+)}{\leq} \varepsilon/2 + p \cdot \varepsilon/2p = \varepsilon \end{aligned}$$

□

En funktion $v: \mathcal{F} \rightarrow \mathbb{R}$, hvor $\forall \varepsilon > 0 \exists \delta > 0 \forall A \in \mathcal{F}, \mu(A) \leq \delta : v(A) \leq \varepsilon$ kaldes μ -kontinuerlig

(3.13.3) giver altså at $v: \mathcal{F} \rightarrow \mathbb{R}_+$, $v(A) = \int_A |f| d\mu$, $A \in \mathcal{F}$, $f \in L_1$ er μ -kontinuerlig

En funktion $v: \mathcal{F} \rightarrow \mathbb{R}$, der opfylder $\forall A \in \mathcal{F} : \mu(A) = 0 \Rightarrow v(A) = 0$ kaldes μ -satureret.

Af (3.13.3) følger, at $v: \mathcal{F} \rightarrow \mathbb{R}_+$, $v(A) = \int_A |f| d\mu$, $A \in \mathcal{F}$, $f \in L_1(\mu)$ er μ -satureret.

3.14 Standardbeviset

$$(\Omega, \mathcal{F}) \text{ m}$$

$$M^+(\Omega, \mathcal{F}) := \{f: \Omega \rightarrow \mathbb{R}_+ \mid f \in \overline{M}(\Omega, \mathcal{F})\}$$

Påstand:

$$L_1, L_2: M^+(\Omega, \mathcal{F}) \rightarrow \mathbb{R}_+$$

$j=1,2$:

$$\forall f, g \in M^+(\Omega, \mathcal{F}), a \geq 0: L_j(f+g) = L_j(f) + L_j(g), L_j(a \cdot f) = a \cdot L_j(f) \quad (1)$$

$$\forall \{f_n\}_{n \geq 1}, f \in M^+(\Omega, \mathcal{F}), f_n \uparrow f: \lim_{n \rightarrow \infty} L_j(f_n) = L_j(f) \quad (2)$$

$$\forall F \in \mathcal{F}: L_1(1_F) = L_2(1_F) \quad (3)$$

Da gælder: $\forall f, g \in M^+(\Omega, \mathcal{F}): L_1(f) = L_2(g)$

Bevis:

Ved simpel induktion med brug af antagelserne (1) og (3) fås at for positive simple funktioner $g, g \in M^+(\Omega, \mathcal{F})$ gælder $L_1(g) = L_2(g)$. af 1.46 (Approximation af målelige funktioner), som siger at for enhver pos. målelig fkt., f eksisterer en følge af simple pos. målelige funktioner, $\{f_n\}_{n \geq 1}: \forall \omega \in \Omega: \lim_{n \rightarrow \infty} f_n(\omega) = f(\omega)$, og antagelse (2), der giver $\lim_{n \rightarrow \infty} L_j(f_n) = L_j(f)$, $j=1,2$ har vi dermed

$$L_1(f) = \lim_{n \rightarrow \infty} L_1(f_n) = \lim_{n \rightarrow \infty} L_2(f_n) = L_2(f)$$

□

3.15 (Den første transformationsætning)

$(\Omega, \mathcal{F}, \mu) \in \mathcal{M}$; $T: \Omega \rightarrow M$ T er $(\mathcal{F}, \mathcal{B})$ -målelig

$\mu_T := \mu \circ T^{-1}$ (dvs. μ_T betegner billedmålet af μ under T)

$g \in \bar{M}(M, \mathcal{B})$

Påstand:

$$\int_M g d\mu_T = \int_{\Omega} (g \circ T) d\mu \quad \left(\begin{array}{l} \text{Hvis det ene integral eksisterer, eksisterer det} \\ \text{andet, og i så tilfælde er de lig hinanden} \end{array} \right)$$

Bevis:

$$L_1, L_2: M^+(M, \mathcal{B}) \rightarrow \bar{\mathbb{R}}_+$$

$$g \in M^+(M, \mathcal{B}): \quad L_1(g) := \int_M g d\mu_T, \quad L_2(g) := \int_{\Omega} (g \circ T) d\mu$$

(Bm. juf. 3.1.3 er L_1, L_2 veldefinerede)

L_1, L_2 opfylder juf. 3.1.8 + 3.1.10 (lin. af int. af pos. fkt.) 3.14.1

Ligeledes opfylder L_1, L_2 3.14.2 juf. 3.1.9

Lad $B \in \mathcal{B}$ være vilkårlig

$$\begin{aligned} L_1(1_B) &= \int_M 1_B d\mu_T \stackrel{3.8.1}{=} \mu_T(B) := \mu(T^{-1}(B)) \stackrel{3.8.1}{=} \int_{\Omega} 1_{T^{-1}(B)} d\mu = \int_{\Omega} 1_B \circ T d\mu \\ &= L_2(1_B) \end{aligned}$$

Dvs. L_1, L_2 opfylder 3.14.3

Af 3.14 (standardbeviset) følger, at $L_1 = L_2$

Idet $g, g^+ \in M^+(M, \mathcal{B})$ (da $g \in \bar{M}(M, \mathcal{B})$) gælder dermed

$$\int_M g^- d\mu_T = L_1(g^-) = L_2(g^-) = \int_{\mathcal{Z}} g^- \circ T d\mu \quad \text{og}$$

$$\int_M g^+ d\mu_T = L_1(g^+) = L_2(g^+) = \int_{\mathcal{Z}} g^+ \circ T d\mu$$

Det ønskede følger af definitionen (3.2.2) på integralet $\square$

Påstand:

$\{a_n\}_{n=1}^\infty \in \mathbb{R}_+$; $\{\mu_n\}_{n=1}^\infty$ betegner mål på $(\Omega, \mathcal{F})$ således : $\mu = \sum_{n=1}^\infty a_n \mu_n$

Da gælder:

$$\forall f \in L(\mu) : \int_{\mathcal{Z}} f d\mu = \sum_{n=1}^\infty a_n \int_{\mathcal{Z}} f d\mu_n$$

Beweis:

$$L_1, L_2 : M^+(\Omega, \mathcal{F})$$

$$g \in M^+(\Omega, \mathcal{F}) : L_1(g) := \int_{\mathcal{Z}} g d\mu, \quad L_2(g) := \sum_{n=1}^\infty a_n \int_{\mathcal{Z}} g d\mu_n$$

(L_1, L_2 er juf. 3.1.3 veldefineret)

L_1, L_2 opfylder 3.14.1-2 juf. 3.1.8-10 (og teorien om rækker af pos. led).

Lad $F \in \mathcal{F}$ være givet

$$L_1(1_F) = \int_{\mathcal{Z}} 1_F d\mu \underset{3.8.1}{=} \mu(F) \underset{\substack{\uparrow \\ \text{pr. antagelse}}}{=} \sum_{n=1}^\infty a_n \cdot \mu_n(F) \underset{3.8.1}{=} \sum_{n=1}^\infty a_n \int_{\mathcal{Z}} 1_F d\mu_n = L_2(1_F)$$

Dvs. L_1, L_2 opfylder 3.14.3

Af 3.14 (standardbrevet) følger, at $L_1 = L_2$

Lad $f \in L(\mu)$ være givet

Idet $f^-, f^+ \in M^+(\Omega, \mathcal{F})$ har vi dermed

$$\int_{\Omega} f^- d\mu = L_1(f^-) = L_2(f^-) = \sum_{n=1}^{\infty} a_n \int_{\Omega} f^- d\mu_n$$

$$\int_{\Omega} f^+ d\mu = L_1(f^+) = L_2(f^+) = \sum_{n=1}^{\infty} a_n \int_{\Omega} f^+ d\mu_n$$

Af def. 3.2.2 følger ($f \in L(\mu)$)

$$\begin{aligned} \int_{\Omega} f d\mu &= \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu = \sum_{n=1}^{\infty} a_n \int_{\Omega} f^+ d\mu_n - \sum_{n=1}^{\infty} a_n \int_{\Omega} f^- d\mu_n \\ &\stackrel{\substack{\uparrow \\ \text{regulærlig for} \\ \text{reeller} \\ \text{(husk den ene er konstant)}}}{=} \sum_{n=1}^{\infty} a_n \left(\int_{\Omega} f^+ d\mu_n - \int_{\Omega} f^- d\mu_n \right) \stackrel{3.2.2}{=} \sum_{n=1}^{\infty} a_n \int_{\Omega} f d\mu_n \end{aligned}$$

□

Definitioner:

$(\Omega, \mathcal{F}, \mu) @m$

Antag $\mu_n, n \geq 1$ betegner endelige mål, hvis $\mu = \sum \mu_n$, siges μ at være sum-endeligt

Hvis $\exists \{F_n\}_{n=1}^{\infty} \in \mathcal{F} : \mu(F_n) < \infty \wedge \bigcup_{n=1}^{\infty} F_n = \Omega$ siges μ at være σ -endeligt.

Hvis $\forall F \in \mathcal{F}, \mu(F) > 0 \exists F_0 \in \mathcal{F} : F_0 \subseteq F \wedge 0 < \mu(F_0) < \infty$, siges μ at være endeligt funderet.

λ på $\mathbb{R}^k, k \geq 1$ er sum-endeligt, σ -endeligt og endeligt funderet.
Tallermålet på en overtællig mængde er endeligt funderet.

3.16 Semi-endelige mål

Påstand:

$(\Omega, \mathcal{F}) @ \mu, \nu$ er mål på $(\Omega, \mathcal{F})$

$\lambda: \mathcal{F} \rightarrow \mathbb{R}_+$, $\lambda(\emptyset) = 0$

Der gælder:

Alle endelige mål er σ -endelige (3.16.1)

Alle σ -endelige mål er sum-endelige og endeligt funderet (3.16.2)

Hvis μ er sum-endelig, da $\exists \xi$, ξ endeligt mål: $N(\mu) = N(\xi)$ (3.16.3)

Hvis μ er sum-endelig og ν er μ -satureret, da er ν sum-endelig (3.16.4)

μ er sum-endelig, hvis μ er ξ -satureret, hvor ξ er et endeligt (el. σ -endeligt, el. sum-endeligt) mål (3.16.5)

Hvis λ er μ -satureret og subadditiv på $\mathcal{F}$ og hvis $\{F_n\}_{n \geq 1} \in \mathcal{F}$, $F_n \downarrow \emptyset$: $\lambda(F_n) \xrightarrow{n \rightarrow \infty} 0$, da er λ μ -kontinueret (3.16.6)

Bevis:

3.16.1

Antag μ endeligt mål. Vælg $\forall n \geq 1$: $F_n = \Omega$. Vi ser $\forall n \geq 1$: $\mu(F_n) = \mu(\Omega) < \infty$ og $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} \Omega = \Omega$, dvs. μ er σ -endeligt.

3.16.2

Antag μ er σ -endeligt, dvs. $\exists \{F_n\}_{n \geq 1} \in \mathcal{F}$: $\bigcup_{n=1}^{\infty} F_n = \Omega \wedge \forall n \geq 1$: $\mu(F_n) < \infty$

Juf. 1.20.3 definerer $\mu_n: \mathcal{F} \rightarrow \mathbb{R}_+$, $\forall F \in \mathcal{F}$: $\mu_n(F) = \mu(F \cap G_n)$, $n \geq 1$

hvor $G_n = F_n \setminus \bigcup_{i=1}^{n-1} F_i$, $n \geq 2$, $G_1 = F_1$ (disjunktionen af $\{F_n\}$), endelige mål på $\mathcal{F}$

idet $\forall n \geq 1, F \in \mathcal{F}$: $\mu_n(F) = \mu(F \cap G_n) \stackrel{1.3.2}{\leq} \mu(G_n) \stackrel{1.3.2}{\leq} \mu(F_n) < \infty$.

Vi ser $\forall F \in \mathcal{F}$: $\mu(F) = \sum_{n=1}^{\infty} \mu_n(F)$, dvs. μ er sum-endeligt (1.3.5)

Lad $F \in \mathcal{F}$; $\mu(F) > 0$ være givet

$0 < \mu(F) \leq \sum_{n=1}^{\infty} \mu(F \cap F_n)$, dvs. $\exists n \geq 1$: $\mu(F \cap F_n) > 0$. Sæt $F_0 := F \cap F_n$.
Idet $0 < \mu(F_0) \leq \mu(F_n) < \infty$; $F_0 \subseteq F$; $F_0 \in \mathcal{F}$ (pga. (nf)-stabilitet) ser vi, at μ er endeligt funderet.

3.16.3:

Antag μ er sum-endelig, dvs. $\{\mu_n\}_{n \geq 1}$ betegner endelige mål således at $\mu = \sum \mu_n$

Definer $\xi: \mathcal{F} \rightarrow \overline{\mathbb{R}}_+$, $\forall F \in \mathcal{F}$: $\xi(F) = \sum_{n=1}^{\infty} \frac{2^{-n}}{1 + \mu_n(\Omega)} \mu_n(F)$

Af 120.4 følger, at ξ er et mål på $(\Omega, \mathcal{F})$

Idet $\xi(\Omega) = \sum_{n=1}^{\infty} \frac{2^{-n}}{1 + \mu_n(\Omega)} \mu_n(\Omega) \leq \sum_{n=1}^{\infty} 2^{-n} = 1$ ser vi, at ξ er et endeligt mål på $(\Omega, \mathcal{F})$

Idet $N \in \mathcal{N}(\mu) \iff \forall n \geq 1: \mu_n(N) = 0 \iff N \in \mathcal{N}(\xi)$ gælder desuden, at $\mathcal{N}(\mu) = \mathcal{N}(\xi)$.

3.16.4:

Antag μ er sum-endelig og ν er μ -satureret

Af 3.16.3 følger, at der eksisterer et endeligt mål ξ således

$\mathcal{N}(\mu) = \mathcal{N}(\xi)$, da ν er μ -satureret, dvs. $\mathcal{N}(\mu) \subseteq \mathcal{N}(\nu)$, gælder dermed at ν er ξ -satureret (Idet $\mathcal{N}(\xi) \subseteq \mathcal{N}(\nu)$).

$\alpha := \sup \{ \xi(F) \mid F \in \mathcal{F}, \nu(F) < \infty \}$ (α veldef. da $\xi < \infty$ og $\emptyset \in \mathcal{F} \wedge \nu(\emptyset) = 0 < \infty$)

Vælg $\{A_n\}_{n \geq 1} \in \mathcal{F}$: $\forall n \geq 1: \nu(A_n) < \infty \wedge \lim_{n \rightarrow \infty} \xi(A_n) = \alpha$

$B_1 := A_1$, $n \geq 2$: $B_n := A_n \setminus \bigcup_{i=1}^{n-1} A_i$, $B = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$

(dvs. $\{B_n\}_{n \geq 1}$ er disjunktionen af $\{A_n\}_{n \geq 1}$)

Lad $F \in \mathcal{F}$ være vilkårlig

Antag $v(F|B) < \infty$

$$v(A_n \cup (F|B)) = v(A_n) + v(F|B) < \infty \quad (v(A_n) < \infty, v(F|B) < \infty)$$

$$\xi(A_n \cup (F|B)) = \xi(A_n) + \xi(F|B) \leq \alpha \quad (\text{pr. def. af } \alpha, \text{ idet } v(A_n \cup (F|B)) < \infty)$$

Idet $\lim_{n \rightarrow \infty} \xi(A_n) = \alpha$ ser vi, at $\xi(F|B) = 0$

Da v er ξ -satureret har vi dermed, at $v(F|B) = 0$

Ovenstående argument gav resultatet $v(F|B) < \infty \Rightarrow \xi(F|B) = 0$

Ved kontraposition har vi således $\xi(F|B) > 0 \Rightarrow v(F|B) = \infty$

$$\text{Vi kan konkludere } v(F|B) = \begin{cases} 0, & \xi(F|B) = 0 \\ \infty, & \xi(F|B) > 0 \end{cases} \quad (*)$$

Definer for $n \geq 1$: $v_{2n}(F) = v(F \cap B_n)$, $v_{2n-1}(F) = \xi(F|B)$

$\forall n \geq 1$: $v_{2n}(F) = v(F \cap B_n) \stackrel{1.2}{\leq} v(A_n) < \infty$ 1.20.3 siger dermed, at

v_{2n} er et endeligt mål på $(\Omega, \mathcal{F})$.

$$\begin{aligned} \sum_{n=1}^{\infty} (v_{2n}(F) + v_{2n-1}(F)) &= \sum_{n=1}^{\infty} v(F \cap B_n) + \sum_{n=1}^{\infty} \xi(F|B) = v(F \cap B) + v(F|B) \\ &\stackrel{1.3.1}{=} v(F) \end{aligned}$$

Dvs. v er sum-endeligt. (Da $v_{2n}, v_{2n-1}, n \geq 1$ er endelige mål)

3.16.5:

Antag at μ er sum-endelig.

Af 3.16.3 følger, at μ er ξ -satureret, hvor ξ er et endeligt mål. (3.16.1-2 siger ξ er σ -endeligt og sum-endeligt)

Antag at ξ er sum-endeligt og at μ er ξ -satureret

ξ er jvf. 3.16.1-2 sum-endeligt af 3.16.4 følger dermed, at μ er sum-endeligt. Idet σ -endelige og endelige mål jvf.

3.16.1-2 er sum-endelige er vi dermed færdige

3.16.6:

Antag λ er μ -satureret og subadditiv på $\mathcal{F}$, samt at

$$\{F_n\}_{n \geq 1} \in \mathcal{F}, F_n \downarrow \emptyset \implies \lim_{n \rightarrow \infty} \lambda(F_n) = 0$$

Bevis ved modstrid:

Antag λ ikke er μ -kontinueret, dvs.

$$\exists \varepsilon_0 > 0 \forall \delta > 0 \exists F' \in \mathcal{F} : \mu(F') \leq \delta \wedge |\lambda(F')| = \lambda(F') > \varepsilon_0$$

Sæt $\forall n \geq 1: \delta_n = 2^{-n}$ og vælg $F_n \in \mathcal{F} : \mu(F_n) \leq 2^{-n} \wedge \lambda(F_n) > \varepsilon_0$

$$F^k := \bigcup_{n=k}^{\infty} F_n, \quad F := \bigcap_{k=1}^{\infty} F^k = \limsup_{n \rightarrow \infty} F_n$$

Idet $\sum_{n=1}^{\infty} \mu(F_n) \leq \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty$ følger af 1.4.6, at $\mu(F) = 0$

Da λ er μ -satureret gælder dermed $\lambda(F) = 0$

$$\varepsilon_0 < \lambda(F_n)_{n \geq 1} \stackrel{\substack{\uparrow \\ \lambda \text{ subadditiv} \\ (1.3.6)}}{\leq} \lambda(F^k)_{k \geq 1} \leq \lambda(F^k \setminus F) + \lambda(F)_{k \geq 1} = \lambda(F^k \setminus F)_{k \geq 1} \quad (*)$$

$$F^k \setminus F = F^k \cap \left(\bigcup_{k=1}^{\infty} (F^k)^c \right) \downarrow \emptyset \stackrel{\text{pr. antagelse}}{\implies} \lim_{k \rightarrow \infty} \lambda(F^k \setminus F) = 0$$

Af vurderingen (*) følger at $\varepsilon_0 = 0$

$\Downarrow$

□

3.17 (Det ubestemte integral)

$(\Omega, \mathcal{F}, \mu) \in \mathcal{M}$, $\varphi \in M^+(\Omega, \mathcal{F})$, $\mathcal{G} \subseteq \mathcal{F}$, $\mathcal{G}$ σ -algebra
 $\nu: \mathcal{G} \rightarrow \overline{\mathbb{R}}_+$, $\forall G \in \mathcal{G}: \nu(G) = \int_G \varphi d\mu$

Påstand: ν er et mål på $(\Omega, \mathcal{G})$

Bevis:

Idet, $\nu(\emptyset) = \int_{\emptyset} \varphi d\mu = 0$, $\nu: \mathcal{G} \rightarrow \overline{\mathbb{R}}_+$ og ν juf. 3.13.2 er σ -additiv, ser vi, at ν er et mål på $(\Omega, \mathcal{G})$. $\square$

Påstand:

$f \in \overline{M}(\Omega, \mathcal{G})$

$\int_{\Omega} f d\nu = \int_{\Omega} (f \cdot \varphi) d\mu$ (Hvis det ene integral eksisterer, eksisterer det andet, og i så tilfælde er de lig hinanden.)

Bevis:

$L_1, L_2: M^+(\Omega, \mathcal{G}) \rightarrow \overline{\mathbb{R}}_+$

$h \in M^+(\Omega, \mathcal{G}): L_1(h) = \int_{\Omega} h d\nu$, $L_2(h) = \int_{\Omega} (h \cdot \varphi) d\mu$

L_1, L_2 opfylder juf. 3.1.8-10 3.14.1-2

Lad $G \in \mathcal{G}$ være givet

$$L_1(1_G) = \int_{\Omega} 1_G d\nu \stackrel{3.8.1}{=} \nu(G) = \int_G \varphi d\mu = \int_{\Omega} (1_G \cdot \varphi) d\mu = L_2(1_G)$$

Af 3.14 (Standardbeviset) følger, at $L_1 = L_2$

Lad $f \in \overline{M}(\Omega, \mathcal{G})$ være vilkårlig, da $f^+, f^- \in M^+(\Omega, \mathcal{G})$ gælder juf. ovenstående:

$$\int_{\Omega} f^+ d\nu = \int_{\Omega} (f^+ \cdot \varphi) d\mu, \int_{\Omega} f^- d\nu = \int_{\Omega} (f^- \cdot \varphi) d\mu$$

Af def. 3.2.2 følger det ønskede dermed. $\square$

Påstand:

$$f, g \in \overline{M}(\mathcal{G}, \mu) \quad , \quad f, g \in L^1(\mu)$$

$$\forall G \in \mathcal{G} : \int_G f d\mu \leq \int_G g d\mu \quad (*)$$

Hvis

$$f \in L^1(\mu) \vee g \in L^1(\mu) \vee (\mu|_{\mathcal{G}} \text{ er endeligt fundert})$$

da gælder

$$f \leq g \quad \mu\text{-n.o.}$$

Beweis:

$$\text{Antag } f \in L^1(\mu) \vee g \in L^1(\mu)$$

$$D = \{f > g\} \quad , \quad D \in \mathcal{G} \quad (\text{betragt en passende sammensat fkt.})$$

$$\left. \begin{array}{l} \forall \omega \in D : 1_D(\omega) \cdot (f(\omega) - g(\omega)) > 0 \xrightarrow{3.3.3} \int_D (f-g) d\mu \geq 0 \\ (*) \Rightarrow \int_D f d\mu - \int_D g d\mu = \int_D (f-g) d\mu \leq 0 \end{array} \right\} \Rightarrow \int_D (f-g) d\mu = 0$$

$\begin{array}{l} f \in L^1(\mu) \\ \vee g \in L^1(\mu) \end{array}$ $\xrightarrow{3.3.12}$

$$\text{Af 3.3.8 følger idet } \int_D (f-g) d\mu = \int_D 1_D \cdot (f-g) d\mu = 0 \quad , \quad \text{at} \\ 1_D \cdot (f-g) = 0 \quad \mu\text{-n.o.} \iff D \in \mathcal{N}(\mu) \quad , \quad \text{dvs. } f \leq g \quad \mu\text{-n.o.}$$

Antag at $\mu|_{\mathcal{G}}$ er endeligt fundert

$$D = \{f > g\} \quad \text{Vi vil vise } D \in \mathcal{N}(\mu) \text{, idet dette medfører } f \leq g \quad \mu\text{-n.o.}$$

Beweis ved modstrid:

$$\text{Antag } \mu(D) > 0$$

Da $\mu|_{\mathcal{G}}$ er endeligt fundert og $D \in \mathcal{G}$ (betragt passende transformation)

$$\exists G \in \mathcal{G} : G \subseteq D \wedge 0 < \mu(G) < \infty$$

$$\forall n \geq 1 : G_n := G \cap \{g \leq n, f \geq -n\} \quad (G_n \in \mathcal{G} \text{ idet } G, \{g \leq n\}, \{f \geq -n\} \in \mathcal{G})$$

Idet $\mu(G_n) \leq \mu(G) < \infty$ har vi følgende vurdering

$$\forall n \geq 1 : \quad -\infty < \underbrace{-n \cdot \mu(G_n)}_{\substack{3.10.1 \text{ } G_n \\ \text{(Jensens Ulighed)}}} \leq \underbrace{\int_{G_n} f d\mu}_{(*) \text{ } G_n} \leq \underbrace{\int_{G_n} g d\mu}_{3.10.1 \text{ } G_n} \leq n \cdot \mu(G_n) < \infty$$

(Jensens Ulighed) $(*) \text{ } G_n$ (Jensens Ulighed)

Dvs.

$$\int_{G_n} f d\mu - \int_{G_n} g d\mu \stackrel{3.3.12}{=} \int_{G_n} (f-g) d\mu \leq 0 \quad (1)$$

$$\text{Men } G_n \subseteq G \subseteq D, \forall \omega \in D: 1_D(\omega) \cdot (f(\omega) - g(\omega)) > 0 \Rightarrow \forall \omega \in G_n: 1_{G_n}(\omega) \cdot (f(\omega) - g(\omega)) > 0 \\ \stackrel{3.3.3}{\Rightarrow} \int_{G_n} (f-g) d\mu \geq 0 \quad (2)$$

$$(1) \wedge (2) \Rightarrow \int_{G_n} (f-g) d\mu = \int_{\mathbb{R}} 1_{G_n} (f-g) d\mu = \int_{\mathbb{R}} |1_{G_n} (f-g)| d\mu = 0 \stackrel{3.3.8}{\Rightarrow} G_n \in \mathcal{N}(\mu)$$

Idet $\forall n \geq 1: G_n \in \mathcal{N}(\mu)$ gælder $\bigcup_{n=1}^{\infty} G_n \in \mathcal{N}(\mu)$ (simpel anvendelse af def. på $\mu, \mathcal{N}(\mu)$)

$\forall \omega \in D: f(\omega) > -\infty \wedge g(\omega) < \infty$, da $\{g \leq n, f \geq -n\} = \{f \leq g, g \leq n\}$, $n \geq 1$
har vi dermed $\bigcup_{n=1}^{\infty} \{g \leq n, f \geq -n\} = \{f \leq g\} \supseteq D \supseteq G$, dvs. $\bigcup_{n=1}^{\infty} G_n = G \in \mathcal{N}(\mu)$

$\sum$

□

Bem. til 3.17:

Hvis $\mathcal{F} = \mathcal{F}$ opnår vi et mål, ν på $(\Omega, \mathcal{F})$, som benævnes det ubestemte μ -integral af φ

Vi indfører følgende notation:

$$d\nu = \varphi d\mu \quad \text{hvis} \quad \forall F \in \mathcal{F}: \nu(F) = \int_F \varphi d\mu$$

Lad μ, ν betegne to mål på $(\Omega, \mathcal{F})$, hvis $\exists \varphi \in H^+(\Omega, \mathcal{F})$: $d\nu = \varphi d\mu$ siges ν at være absolut μ -kontinuerligt, og vi skriver $\nu \ll \mu$, fkt. φ kaldes en μ -tæthed for ν , og vi indfører notationen

$$\frac{d\nu}{d\mu} = \varphi \quad (\text{dvs. } d\nu = \frac{d\nu}{d\mu} d\mu, \text{ hvis } \nu \ll \mu)$$

Indsættes dette i 3.17.2 fås $\int_{\Omega} f d\nu = \int_{\Omega} (f \frac{d\nu}{d\mu}) d\mu$ (se bem. til 3.17.2)

Påstand: μ er endeligt fundert $\wedge \varphi, \varphi'$ μ -tæth. f. $\nu \Rightarrow \varphi = \varphi'$ μ -n.o.

Bevis:

$$\varphi, \varphi' \text{ } \mu\text{-tætheder for } \nu \Rightarrow \int_{\Omega} \varphi d\mu = \int_{\Omega} \varphi' d\mu$$

Da μ er endeligt fundert følger dermed af 3.17.3, at $\varphi = \varphi'$ μ -n.o. $\square$

μ -tæthed for ν er ikke entydigt bestemt; lad nemlig φ betegne en μ -tæthed og antag $\psi \in H^+(\Omega, \mathcal{F})$, $\varphi = \psi$ μ -n.o.

Af 3.3.4 følger, at $\forall F \in \mathcal{F}: \nu(F) = \int_{\Omega} 1_F \varphi d\mu = \int_{\Omega} 1_F \psi d\mu$, dvs. ψ er en μ -tæthed for ν .

Lad μ, ν betegne to vilkårlige mål på $(\Omega, \mathcal{F})$

Påstand:

$$\nu \text{ } \mu\text{-kontinuerligt} \Rightarrow \nu \text{ er } \mu\text{-satureret} \quad (3.17.8)$$

$$\nu \ll \mu \Rightarrow \nu \text{ er } \mu\text{-satureret} \quad (3.17.9)$$

$$\nu \ll \mu \wedge \nu \text{ er endeligt} \Rightarrow \nu \text{ er } \mu\text{-kontinuerligt} \quad (3.17.10)$$

Bevis:

3.17.8:

Antag ν er μ -kontinuerligt, dvs. $\forall \varepsilon > 0 \exists \delta > 0 \forall A \in \mathcal{F} : \mu(A) \leq \delta \Rightarrow \nu(A) \leq \varepsilon$

Lad $N \in \mathcal{N}(\mu)$ være vilkårlig $\forall \varepsilon > 0 : \mu(N) = 0 \leq \delta \Rightarrow \forall \varepsilon > 0 : \nu(N) \leq \varepsilon$
 $\Rightarrow \nu(N) = 0$ (ses meget nemt v. modstridsbevis)

3.17.9:

Antag $\nu \ll \mu$, dvs. $\exists \varphi \in H^+(\Omega, \mathcal{F}) ; \forall F \in \mathcal{F} : \nu(F) = \int_F \varphi d\mu$

Lad $N \in \mathcal{N}(\mu)$ være vilkårlig, $\nu(N) = \int_N \varphi d\mu = \int_{\Omega} 1_N \varphi d\mu = 0$ jvf. 3.3.8

3.17.10:

$\nu \ll \mu$ $\wedge$ ν er endeligt

$\nu \ll \mu \Rightarrow \nu$ er μ -satureret jvf. 3.17.9

ν er et mål på $(\Omega, \mathcal{F})$, specielt gælder dermed, at ν er subadditiv på $\mathcal{F}$.

Lad $\{F_n\}_{n \in \mathbb{N}} \in \mathcal{F} : F_n \downarrow \emptyset$, $\nu(F_1) \stackrel{1.3.2}{\leq} \nu(\Omega) < \infty \stackrel{1.4.3}{\Rightarrow} \lim_{n \rightarrow \infty} \nu(F_n) = \nu(\emptyset) = 0$,
idet ν er endeligt. 3.16.6 giver dermed det ønskede $\square$

3.20 Produktmål

$(S, \mathcal{A}, \mu), (T, \mathcal{B}, \nu)$ m

Antag at der eksisterer følger $\{\mu_i\}_{i=1}^\infty, \{\nu_j\}_{j=1}^\infty$ af endelige mål på hh. $(S, \mathcal{A})$ og $(T, \mathcal{B})$ således $\mu = \sum_{i=1}^\infty \mu_i, \nu = \sum_{j=1}^\infty \nu_j$ (μ, ν er sum-endelige)

Vi vil nu finde et mål på $(S \times T, \mathcal{A} \otimes \mathcal{B})$

$\forall n, k \in \mathbb{N}: V_{nk} := \{f \in M^+(S \times T, \mathcal{A} \otimes \mathcal{B}) \mid \int_T f(\cdot, t) d\nu_k \in M^+(S, \mathcal{A}), \int_S f(s, \cdot) d\mu_n \in M^+(T, \mathcal{B})\}$

Lad $n, k \in \mathbb{N}$ være vilkårlige

$\forall f, g \in V_{nk}, a \in \mathbb{R}_+: f+g, a \cdot f \in M^+(S \times T, \mathcal{A} \otimes \mathcal{B})$ (Jvf. 1.43.6-7)

og $\int_T f(\cdot, t) d\nu_k, \int_T g(\cdot, t) d\nu_k \in M^+(S, \mathcal{A}) \xRightarrow{1.43.7 + 3.1.8} \int_T (f(\cdot, t) + g(\cdot, t)) d\nu_k \in M^+(S, \mathcal{A})$

$\int_S f(s, \cdot) d\mu_n, \int_S g(s, \cdot) d\mu_n \in M^+(T, \mathcal{B}) \xRightarrow{} \int_S (f(s, \cdot) + g(s, \cdot)) d\mu_n \in M^+(T, \mathcal{B})$

$\int_T f(\cdot, t) d\nu_k \in M^+(S, \mathcal{A}), \int_S f(s, \cdot) d\mu_n \in M^+(T, \mathcal{B}) \xRightarrow{1.43.7 + 3.1.10} \int a \cdot f(\cdot, t) d\nu_k \in M^+(S, \mathcal{A}), \int a \cdot f(s, \cdot) d\mu_n \in M^+(T, \mathcal{B})$

Dvs. $f+g, a \cdot f \in V_{nk}$

Lad $\{f_\ell\}_{\ell=1}^\infty \in V_{nk}, f \in M^+(S \times T, \mathcal{A} \otimes \mathcal{B}): f_\ell \uparrow f$ være givet

Idet $f \in M^+(S \times T, \mathcal{A} \otimes \mathcal{B}) \Rightarrow f(\cdot, t) \in M^+(S, \mathcal{A}), f(s, \cdot) \in M^+(T, \mathcal{B})$ og

$\forall \ell \geq 1: f_\ell \in V_{nk} \Rightarrow f_\ell \in M^+(S \times T, \mathcal{A} \otimes \mathcal{B}) \Rightarrow f_\ell(\cdot, t) \in M^+(S, \mathcal{A}), f_\ell(s, \cdot) \in M^+(T, \mathcal{B})$ og

$f_\ell \uparrow f \Rightarrow f_\ell(\cdot, t) \uparrow f(\cdot, t), f_\ell(s, \cdot) \uparrow f(s, \cdot)$ følger af 1.40.4 og 3.1.9, at

$\int_T f(\cdot, t) d\nu_k = \lim_{\ell \rightarrow \infty} \int_T f_\ell(\cdot, t) d\nu_k \in M^+(S, \mathcal{A}), \int_S f(s, \cdot) d\mu_n = \lim_{\ell \rightarrow \infty} \int_S f_\ell(s, \cdot) d\mu_n \in M^+(T, \mathcal{B})$

Dvs. $f \in V_{nk}$

Denmed har vi vist, at V_{nk} er en $(\uparrow)$ -stabil konvex kegle, dvs. opfylder 1.49.1-2.

Lad $A \times B \subseteq S \times T$ være vilkårlig, antag $\forall a \geq 0: a + 1_{A \times B} \in V_{nk}$

Lad $a > 0$ være givet

Idet $a + 1_{A \times B} \in V_{nk}$ har vi specielt, at $a + 1_{A \times B} \in H^+(S \times T, \mathcal{U} \otimes B)$
 $a \in H^+(S \times T, \mathcal{U} \otimes B)$ (trivielt) af 1.43.7 følger (idet $a + 1_{A \times B} - a = 0$),
 at $1_{A \times B} = 1_A \cdot 1_B \in H^+(S \times T, \mathcal{U} \otimes B)$ (*)

Idet $a + 1_{A \times B} \in V_{nk}$ gælder

$$\int_T (a + 1_{A \times B}) d\mu_E \in M^+(S, \mathcal{U}), \quad \int_S (a + 1_{A \times B}) d\mu_n \in M^+(T, \mathcal{B})$$

|| 3.1.8 + (*) || 3.1.8 + (*)

$$\left(\int_T \alpha dr_k + \int_T 1_{A \times B} dr_k\right) \in M^+(S, \mathcal{A}), \quad \left(\int_S \alpha d\mu_n + \int_S 1_{A \times B} d\mu_n\right) \in M^+(T, \mathcal{B}) \quad (+)$$

Idet $\int_T a \, d\nu_k = a \cdot \nu_k(T) < \infty$, $\int_S a \, d\mu_n = a \cdot \mu_n(S) < \infty$ (ν_k, μ_n er endelige
(og situationen $\infty - \infty$ derfor ikke optræder) følger, da (†) og

$$\int_T \text{adv}_k = a \cdot v_k(T) \in M^+(S, \mathcal{A}), \int_S \text{adv}_m = a \cdot \mu_m(S) \in M^+(T, \mathcal{B}), \text{ of } 1.437, \text{ at}$$

$$\int_T 1_{A \times B} dv_k = \int_T a dv_k + \int_T 1_{A \times B} dv_k - \int_T a dv_k \in M^+(S, \mu) \quad \text{og}$$

$$\int_S 1_{A \times B} d\mu = \int_S a d\mu + \int_S 1_{A \times B} d\mu - \int_S a d\mu \in M^+(T, \mathcal{B})$$

Dus. $1_{A \times B} \in V_{nk}$

Da $1_{(A \times B)^c} = 1 - 1_{A \times B} \geq 0$, $1 \in M^+(S \times T, \mathcal{A} \otimes \mathcal{B})$ og $1_{A \times B} \in M^+(S \times T, \mathcal{A} \otimes \mathcal{B})$ (jf. *)
 følger af 1.43.7, at $1_{(A \times B)^c} \in M^+(S \times T, \mathcal{A} \otimes \mathcal{B})$

$$\int_T 1 dv_k = v_k(\pi) \in M^+(S, \mathcal{A}), \quad \int_S 1 d\mu_n = \mu_n(S) \in M^+(T, \mathcal{B}) \quad (\text{twice})$$

$$1_{A \times B} \leq 1 \xRightarrow{3.1.4} \infty > \int_T 1 d\nu_k \geq \int_T 1_{A \times B} d\nu_k, \quad \infty > \int_S 1 d\mu_n \geq \int_S 1_{A \times B} d\mu_n$$

$$\Downarrow 1_{A \times B} \in V_{nk}$$

$$\int_T 1_{(A \times B)^c} dv_k = \int_T (1 - 1_{A \times B}) dv_k \stackrel{5.3.11-12}{=} \int_T 1 dv_k - \int_T 1_{A \times B} dv_k \in M^+(S, \mathcal{A})$$

$$\int_S 1_{(A \times B)^c} d\mu_n = \int_S (1 - 1_{A \times B}) d\mu_n \stackrel{3.3.11-12}{=} \int_S 1 d\mu_n - \int_S 1_{A \times B} d\mu_n \in M^+(T, \mathcal{B})$$

Dus $1_{(A \times B)^c} \in V_{nk}$

Vi har dermed vist, at V_{nk} opfylder 1.49.3

Definer $\mathcal{G} := \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$, $W := \{1_G : S \times T \rightarrow \mathbb{R}_+ \mid G \in \mathcal{G}\}$

$\forall A \times B, A' \times B' \in \mathcal{G} : A \times B \cap A' \times B' = (A \cap A') \times (B \cap B')$ da $\mathcal{A}, \mathcal{B}$ er σ -algebra (og dermed (nf)-stabile) følger, at $\mathcal{G}$ er (nf)-stabil. juf. def. på $\mathcal{G}$.

$\forall f, g \in W \exists A \times B, A' \times B' \in \mathcal{G} : 1_{A \times B} = f, 1_{A' \times B'} = g$

$f \cdot g = 1_{A \times B} \cdot 1_{A' \times B'} = 1_{(A \times B) \cap (A' \times B')} \in W$ (idet $\mathcal{G}$ er (nf)-stabil)

Da $W \subseteq \text{con}_r(W)$ (pr. def. af $\text{con}_r(W)$ som mindste $\uparrow$ -stabile konvekse kgl. indh. W) ser vi, at W, V_{nk} opfylder 1.49.4 ($W \subseteq V_{nk}$ hvilket er klart ud fra def.)

Lad $g \in W$ være vilkårlig

$\exists A \times B \in \mathcal{G} : g = 1_{A \times B}$, $\forall w \in S \times T : 1_{A \times B}(w) \leq 1$

Da $\mathcal{A}, \mathcal{B}$ er σ -algebra gælder $A^c, S \in \mathcal{A}$, $B, B^c \in \mathcal{B}$, dvs.

juf. def. på $\mathcal{G}$, at $1_{A^c \times B}, 1_{S \times B^c} \in W$ og dermed juf. def. på $\text{con}_r(W)$

$1 - 1_{A \times B} = 1_{A^c \times B} + 1_{S \times B^c} \in \text{con}_r(W)$.

Dermed har vi vist, at W, V_{nk} opfylder 1.49.5

$$\mathcal{A} \otimes \mathcal{B} \stackrel{1.43.2}{=} \sigma\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} = \sigma(\mathcal{G}) \stackrel{\text{def. af } \mathcal{G}}{=} \sigma(W) \quad *$$

$$*: \sigma(\mathcal{G}) = \sigma\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} = \sigma\{1_{A \times B} \mid A \in \mathcal{A}, B \in \mathcal{B}\} = \sigma(W)$$

$\uparrow$
=: def. af målelighed
og mindste σ -alg.
=: ——— || ———

Definer for alle $f \in Y_{nk}$ $L_{1nk}, L_{2nk}: Y_{nk} \rightarrow \bar{\mathbb{R}}_+$,

$$L_{1nk}(f) := \int_S d\mu_n \int_T f d\nu_k, \quad L_{2nk} := \int_T d\nu_k \int_S f d\mu_n$$

Pr. def. af Y_{nk} ser vi $L_{1nk}, L_{2nk}: M^+(S \times T, \mathcal{A} \otimes \mathcal{B}) \rightarrow \bar{\mathbb{R}}_+$

Lad $f, g \in Y_{nk}$, $a \geq 0$ være vilkårlig

$$\begin{aligned} L_{1nk}(f+g) &= \int_S d\mu_n \int_T (f+g) d\nu_k = \int_S d\mu_n \left\{ \int_T f d\nu_k + \int_T g d\nu_k \right\} \\ &\quad \uparrow \text{def. af } Y_{nk} + 3.1.8 (1.43.7) \\ &= \int_S \left\{ \int_T f d\nu_k \right\} d\mu_n + \int_S \left\{ \int_T g d\nu_k \right\} d\mu_n = L_{1nk}(f) + L_{1nk}(g) \end{aligned}$$

$$L_{1nk}(a \cdot f) = \int_S d\mu_n \int_T a \cdot f d\nu_k \stackrel{\text{def. af } Y_{nk} + 3.1.10}{=} \int_S d\mu_n a \cdot \int_T f d\nu_k \stackrel{\uparrow}{=} a \cdot \int_S d\mu_n \int_T f d\nu_k = a \cdot L_{1nk}(f)$$

Tilsvarende for L_{2nk} , dvs. L_{1nk}, L_{2nk} opfylder 3.14.1

Lad $\{f_n\}_{n \in \mathbb{N}}$, $f \in Y_{nk}$, $f_n \uparrow f$

$$\lim_{n \rightarrow \infty} L_{1nk}(f_n) = \lim_{n \rightarrow \infty} \int_S d\mu_n \int_T f_n d\nu_k \stackrel{\text{def. af } Y_{nk} + 3.1.4 \Rightarrow \int_T f_n d\nu_k \uparrow \int_T f d\nu_k}{=} \int_S d\mu_n \int_T f d\nu_k = L_{1nk}(f)$$

+ 3.1.9

Tilsvarende for L_{2nk} , dvs. L_{1nk}, L_{2nk} opfylder 3.14.2

Lad $A \times B \in S \times T$ være vilkårlig

Antag $\forall a > 0$: $a + 1_{A \times B} \in V_{nk}$ og $L_1(a + 1_{A \times B}) = L_2(a + 1_{A \times B})$ (*)

Da V_{nk} opfylder 1.49.3 (till. vist) gælder $1_{A \times B}, 1_{(A \times B)^c} \in V_{nk}$

Lad $a > 0$ være givet

At $a \in V_{nk}$ er trivielt (se def. på mætlige fkt. + 3.8.1 + def. af V_{nk})

Desuden gælder

$$L_{1nk}(a) := \int_S d\mu_n \int_T a dv_k \underset{\substack{\uparrow \\ 3.1.10}}{=} \int_S \left\{ a \cdot \int_T dv_k \right\} d\mu_n \underset{\substack{\uparrow \\ 3.8.1}}{=} \int_S a \cdot v_k(T) d\mu_n \underset{\substack{\uparrow \\ 3.1.10 \\ + 3.8.1}}{=} a \cdot v_k(T) \cdot \mu_n(S)$$

$$\underset{\substack{\uparrow \\ 3.8.1}}{=} \int_T a \cdot \mu_n(S) dv_k \underset{\substack{\uparrow \\ 3.8.1 \\ + 3.1.10}}{=} \int_T dv_k \int_S a d\mu_n =: L_{2nk}(a) \quad (*)$$

$$\begin{aligned} L_{1nk}(1_{A \times B}) &= L_{1nk}(a + 1_{A \times B} - a) := \int_S d\mu_n \int_T (a + 1_{A \times B} - a) dv_k \\ &\underset{\substack{\uparrow \\ 2 \times 3.3.12 \\ + 3.3.11}}{=} \int_S d\mu_n \int_T (a + 1_{A \times B}) dv_k - \int_S d\mu_n \int_T a dv_k = L_{1nk}(a + 1_{A \times B}) - L_{1nk}(a) \end{aligned}$$

$$\begin{aligned} &\underset{(*)}{=} L_{2nk}(a + 1_{A \times B}) - L_{2nk}(a) = \int_T dv_k \int_S (a + 1_{A \times B}) d\mu_n - \int_T dv_k \int_S a d\mu_n \\ &\underset{\substack{\uparrow \\ 3.3.11 \\ + 2 \times 3.3.12}}{=} \int_T dv_k \int_S (a + 1_{A \times B} - a) d\mu_n =: L_{2nk}(a + 1_{A \times B} - a) = L_{2nk}(1_{A \times B}) \end{aligned}$$

$$\begin{aligned} L_{1nk}(1_{(A \times B)^c}) &\underset{\substack{\uparrow \\ 1_{(A \times B)^c} = 1 - 1_{A \times B}}}{=} L_{1nk}(1 - 1_{A \times B}) := \int_S d\mu_n \int_T (1 - 1_{A \times B}) dv_k \\ &\underset{\substack{\uparrow \\ 2 \times 3.3.12 \\ + 3.3.11}}{=} \int_S d\mu_n \int_T 1 dv_k - \int_S d\mu_n \int_T 1_{A \times B} dv_k = L_{1nk}(1) - L_{1nk}(1_{A \times B}) \end{aligned}$$

$$\underset{(*)}{=} L_{2nk}(1) - L_{2nk}(1_{A \times B}) = \int_T dv_k \int_S 1 d\mu_n - \int_T dv_k \int_S 1_{A \times B} d\mu_n$$

$$\underset{\substack{\uparrow \\ 3.3.11 \\ + 2 \times 3.3.12}}{=} \int_T dv_k \int_S (1 - 1_{A \times B}) d\mu_n =: L_{2nk}(1 - 1_{A \times B}) \underset{\substack{\uparrow \\ 1_{(A \times B)^c} = 1 - 1_{A \times B}}}{=} L_{2nk}(1_{(A \times B)^c})$$

Dvs. L_{1nk}, L_{2nk} opfylder 3.14.5

Lad $g \in \mathcal{W}$ være vilkårlig

Dvs. $\exists A \in \mathcal{A}, B \in \mathcal{B} : 1_{A \times B} = g$

$$L_{1nk}(g) = L_{1nk}(1_{A \times B}) := \int_S d\mu_n \int_T 1_{A \times B} d\nu_k \stackrel{1_{A \times B} = 1_A \cdot 1_B}{=} \int_S d\mu_n \int_T 1_A \cdot 1_B d\nu_k$$

$$\stackrel{\substack{3.3.11 \ S \\ + B \in \mathcal{B} + 3.8.1}}{=} \int_S 1_A \cdot \nu_k(B) d\mu_n = \mu_n(A) \cdot \nu_k(B) \stackrel{\substack{3.3.11 \\ + A \in \mathcal{A} + 3.8.1}}{=} \int_T 1_B \cdot \mu_n(A) d\nu_k \stackrel{\substack{3.3.11 \ T \\ + 3.8.1}}{=} \int_T d\nu_k \int_S 1_A \cdot 1_B d\mu_n$$

$$\stackrel{\substack{\uparrow \ T \\ 1_{A \times B} = 1_A \cdot 1_B}}{=} \int_T d\nu_k \int_S 1_{A \times B} d\mu_n =: L_{2nk}(1_{A \times B}) = L_{2nk}(g)$$

Dvs. L_{1nk}, L_{2nk} opfylder 3.14.6

Af ovenstående følger jf. 'Standardbeviset' (3.14), at $\forall n, k \in \mathbb{N}$:

$$M^+(S \times T, \mathcal{A} \otimes \mathcal{B}) \subseteq V_{nk} \text{ og } \forall f \in M^+(S \times T, \mathcal{A} \otimes \mathcal{B}) : L_{1nk}(f) = L_{2nk}(f)$$

Idet $V_{nk} \subseteq M^+(S \times T, \mathcal{A} \otimes \mathcal{B})$ pr. def. fås dermed $M^+(S \times T, \mathcal{A} \otimes \mathcal{B}) = V_{nk}$ og $\forall f \in M^+(S \times T, \mathcal{A} \otimes \mathcal{B}) : L_{1nk}(f) = L_{2nk}(f)$ for vilk. $n, k \in \mathbb{N}$.

Af 3.15.3 (1. Transformationssetn.) følger $M^+(S \times T, \mathcal{A} \otimes \mathcal{B}) \subseteq L(\mu) \cap L(\nu)$

$$\int_S f d\mu = \sum_{n=1}^{\infty} \int_S f d\mu_n \in M^+(T, \mathcal{B}) \text{ (da } \forall n \geq 1 : \int_S f d\mu_n \in M^+(T, \mathcal{B}) + 1.43.7)$$

$$\int_T f d\nu = \sum_{k=1}^{\infty} \int_T f d\nu_k \in M^+(S, \mathcal{A}) \text{ (da } \forall k \geq 1 : \int_T f d\nu_k \in M^+(S, \mathcal{A}) + 1.43.7),$$

for alle $f \in M^+(S \times T, \mathcal{A} \otimes \mathcal{B})$

Desuden fås

$$\begin{aligned}
 \int_S d\mu \int_T f \, dv &= \int_S \left(\sum_{k=1}^{\infty} \int_T f \, dv_k \right) d\mu = \sum_{n=1}^{\infty} \int_S \left(\sum_{k=1}^{\infty} \int_T f \, dv_k \right) d\mu_n \\
 &\stackrel{3.7.4}{=} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int_S d\mu_n \int_T f \, dv_k = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} L_{1nk}(f) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} L_{2nk}(f) \\
 &\stackrel{\substack{\text{(da leddene i summen er positive)}}}{=} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} L_{2nk}(f) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_T dv_k \int_S f \, d\mu_n \stackrel{3.7.4}{=} \sum_{k=1}^{\infty} \int_T \left(\sum_{n=1}^{\infty} \int_S f \, d\mu_n \right) dv_k \\
 &= \sum_{k=1}^{\infty} \int_T \left(\int_S f \, d\mu \right) dv_k = \int_T dv \int_S f \, d\mu \quad \text{for alle } f \in M^+(S \times T, \mathcal{A} \otimes \mathcal{B})
 \end{aligned}$$

Definer $\mu \otimes v: \mathcal{A} \otimes \mathcal{B} \rightarrow \overline{\mathbb{R}}_+$, $\forall F \in \mathcal{A} \otimes \mathcal{B}: \mu \otimes v(F) := \int_S d\mu \int_T 1_F \, dv = \int_T dv \int_S 1_F \, d\mu$
 (her betegnes produktmålet af μ og v på $\mathcal{A} \otimes \mathcal{B}$)
 (Tallies Setn.)

$$\mu \otimes v(\emptyset) = \int_S d\mu \int_T 1_{\emptyset \times \emptyset} \, dv = \int_S d\mu \int_T 1_{\emptyset} \cdot 1_{\emptyset} \, dv \stackrel{3.3.11}{=} \int_S 1_{\emptyset} \cdot 0 \, d\mu \stackrel{3.3.11}{=} 0 \cdot 0 = 0$$

Dvs. $\mu \otimes v$ er en mængdefunktion, der opfylder 1.1.4

Lad $\{F_n\}_{n=1}^{\infty} \in \mathcal{A} \otimes \mathcal{B}$, $\forall n \neq m: F_n \cap F_m = \emptyset$ betegne en vilk. følge (*)

$$\begin{aligned}
 \mu \otimes v\left(\bigcup_{n=1}^{\infty} F_n\right) &\stackrel{(*)}{=} \int_S d\mu \int_T 1_{\bigcup_{n=1}^{\infty} F_n} \, dv = \int_S d\mu \int_T \sum_{n=1}^{\infty} 1_{F_n} \, dv \stackrel{3.7.4}{=} \int_S \left(\sum_{n=1}^{\infty} \int_T 1_{F_n} \, dv \right) d\mu \\
 &\stackrel{3.7.4}{=} \sum_{n=1}^{\infty} \int_S d\mu \int_T 1_{F_n} \, dv = \sum_{n=1}^{\infty} \mu \otimes v(F_n), \quad \text{dvs. } \mu \otimes v \text{ opfylder 1.1.5}
 \end{aligned}$$

Dermed har vi vist, at $\mu \otimes v$ er et mål på $\mathcal{A} \otimes \mathcal{B}$

Antag $A \in \mathcal{A}$, $B \in \mathcal{B}$, $A \times B \in \mathcal{A} \otimes \mathcal{B}$ juf. 1.43.2

$$\mu \otimes \nu(A \times B) := \int_S d\mu \int_T 1_{A \times B} d\nu = \int_S d\mu \int_T 1_A \cdot 1_B d\nu = \mu(A) \cdot \nu(B)$$

Pastand:

Antag μ, ν er σ -endelige, og λ er et mål på $\mathcal{A} \otimes \mathcal{B}$,
der opfylder: $\forall A \in \mathcal{A}, B \in \mathcal{B} : \lambda(A \times B) = \mu(A) \nu(B)$

Da gælder $\lambda = \mu \text{ or}$

Bevi's:

Det er klart, at $\lambda|_g = \mu \otimes \lambda|_g$ (husk def. på g , og at g er (nf) stb.)

Da μ, v er σ -endelige gælder:

$$\exists \{G_n\}_{n \geq 1} \in \mathcal{A}, \{H_n\}_{n \geq 1} \in \mathcal{B}, \bigcup_{n=1}^{\infty} G_n = S, \bigcup_{n=1}^{\infty} H_n = T, \forall n \geq 1: \mu(G_n) < \infty \wedge \nu(H_n) < \infty$$

Def. $\forall n \geq 1: F_n := G_n \times H_n, \{F_n\}_{n \in \mathbb{N}}$

Viser, at $S \times T = \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} G_n \times H_n = \left(\bigcup_{n=1}^{\infty} G_n \right) \times \left(\bigcup_{n=1}^{\infty} H_n \right)$

og $n \geq 1$: $\lambda(F_n) = \mu \vee (F_n)$ idet $\{F_n\}_{n \geq 1} \in \mathcal{G}$

Af sætn. om entydighed af mål fås $\lambda = \mu$ over



3.22 Konvergens i q -mean

$$(\Omega, \mathcal{F}, \mu) \text{ @ } , f, g, \{f_n\}_{n \geq 1} \in \bar{M}(\Omega, \mathcal{F})$$

$$f_n \xrightarrow{n \rightarrow \infty} f : \forall \omega \in \Omega \quad \forall \varepsilon > 0 \exists N \geq 1 \forall n \geq N : |f_n(\omega) - f(\omega)| < \varepsilon$$

$$f_n \xrightarrow{n \rightarrow \infty} f : \forall \varepsilon > 0 \exists N \geq 1 \forall \omega \in \Omega \forall n \geq N : |f_n(\omega) - f(\omega)| < \varepsilon$$

$$f_n \xrightarrow{n \rightarrow \infty} f \text{ } \mu\text{-n.o.} : \exists \bar{N} \in \mathcal{N}(\mu) \forall \omega \in \Omega \setminus \bar{N} \quad \forall \varepsilon > 0 \exists N \geq 1 \forall n \geq N : |f_n(\omega) - f(\omega)| < \varepsilon$$

$$f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \mu\text{-mål} : \forall \varepsilon > 0 \exists N \geq 1 \forall n \geq N : \mu(|f_n - f| > \varepsilon) < \varepsilon \quad (\infty - \infty := 0)$$

$$(f_n \xrightarrow{n \rightarrow \infty} f \text{ } \mu\text{-n.o.} : \exists \bar{N} \in \mathcal{N}(\mu) \forall \varepsilon > 0 \exists N \geq 1 \forall \omega \in \Omega \setminus \bar{N} \forall n \geq N : |f_n(\omega) - f(\omega)| < \varepsilon)$$

$$\mathcal{L}(0, \infty) := \{\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \mid \varphi \text{ voks., v. kont., } \forall x > 0 : \varphi(x) > 0, \varphi(0) = \lim_{x \downarrow 0} \varphi(x) = 0\}$$

Lad $\varphi \in \mathcal{L}(0, \infty)$ være vilkårlig

$$\|f\|_\varphi := \int_\Omega \varphi(|f|) d\mu$$

$$f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \varphi\text{-mean} : \forall \varepsilon > 0 \exists N \geq 1 \forall n \geq N : \|f_n - f\|_\varphi < \varepsilon$$

(φ -mean = φ -middel)

Påstand: $f_n \xrightarrow{n \rightarrow \infty} f$ i φ -mean $\Rightarrow f_n \xrightarrow{n \rightarrow \infty} f$ i μ -mål (*)

Beweis:

Lad $\varepsilon > 0$ være vilkårlig

Da $f_n \xrightarrow{n \rightarrow \infty} f$ i φ -mean $\exists N \geq 1 \forall n \geq N: \|f_n - f\|_{\varphi} \leq \frac{\varphi(\varepsilon) \cdot \varepsilon}{2}$, dermed $\forall n \geq N$

$$\mu(|f_n - f| > \varepsilon) \leq \mu(|f_n - f| \geq \varepsilon) \stackrel{\substack{\uparrow \\ 3.9 \text{ Markov's Uglighed}}}{\leq} \frac{1}{\varphi(\varepsilon)} \int_{\mathbb{R}} \varphi(|f_n - f|) d\mu = \frac{1}{\varphi(\varepsilon)} \|f_n - f\|_{\varphi} \leq \frac{\varepsilon}{2} < \varepsilon$$

Dvs. $\forall \varepsilon > 0 \exists N \geq 1 \forall n \geq N: \mu(|f_n - f| > \varepsilon) < \varepsilon$

□

Påstand: Hvis φ er konvex: $\|f + g\|_{\varphi} \leq \|f\|_{\varphi} + \|g\|_{\varphi}$

Beweis:

Dette er en direkte konsekvens af Minkowski's Ulighed og def. på $\|\cdot\|_{\varphi}$

□

Påstand:

Hvis φ er begrænset og $\exists b > 0: \sup_{n \geq 1} \{\mu(0 < |f_n - f| \leq b)\} < \infty$:

$$f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \varphi\text{-mean} \iff f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \mu\text{-mål}$$

Beweis:

' $\Rightarrow$ ': Er indholdet af (*)

' $\Leftarrow$ ':

Vi skal vise, at $\lim_{n \rightarrow \infty} \|f_n - f\|_{\varphi} = 0$

$$\text{Vurdering: } \|f_n - f\|_{\varphi} := \int_{\mathbb{R}} \varphi(|f_n - f|) d\mu$$

$$\stackrel{\substack{3.8.1 \\ 3.3.12}}{\uparrow} = \int_{\mathbb{R}} \varphi(|f_n - f|) \cdot 1_{\{|f_n - f| = 0\}} d\mu + \int_{\mathbb{R}} \varphi(|f_n - f|) \cdot 1_{\{0 < |f_n - f| \leq \varepsilon\}} d\mu + \int_{\mathbb{R}} \varphi(|f_n - f|) \cdot 1_{\{|f_n - f| > \varepsilon\}} d\mu$$

$0 < \varepsilon \leq b$

$$\stackrel{\substack{3.3.3 \\ + 3.8.1}}{\leq} \varphi(\varepsilon) \cdot \mu(0 < |f_n - f| \leq \varepsilon) + a \cdot \mu(|f_n - f| > \varepsilon)$$

↳ idet $\exists a > 0 \forall x > 0: \varphi(x) \leq a$ (φ begrænset) og $\varphi(|f_n - f|)|_{\{|f_n - f| = 0\}} = 0$
 samt $\varphi(|f_n - f|)|_{\{0 < |f_n - f| \leq \varepsilon\}} \leq \varphi(\varepsilon)$ (idet φ er vokse.)

$$\stackrel{\varepsilon \leq b}{\leq} \varphi(\varepsilon) \cdot \mu(0 < |f_n - f| \leq b) + a \cdot \mu(|f_n - f| > \varepsilon)$$

$$\leq \varphi(\varepsilon) \cdot c + a \cdot \mu(|f_n - f| > \varepsilon)$$

$\sup_{n \geq 1} \{\mu(0 < |f_n - f| \leq b)\} < \infty$
 $\Rightarrow \exists c > 0 \forall n \geq 1: \mu(0 < |f_n - f| \leq b) \leq c$

⇓

$$\begin{aligned} |f_n - f|_{\varphi} &\stackrel{\substack{\uparrow \\ |f_n - f|_{\varphi} \text{ afh. af } \varepsilon}}{=} \lim_{\varepsilon \downarrow 0} (|f_n - f|_{\varphi}) \leq \lim_{\varepsilon \downarrow 0} (\varphi(\varepsilon) \cdot c + a \cdot \mu(|f_n - f| > \varepsilon)) \\ &= c \cdot \lim_{\varepsilon \downarrow 0} \varphi(\varepsilon) + a \cdot \lim_{\varepsilon \downarrow 0} \mu(|f_n - f| > \varepsilon) \\ &= a \cdot \lim_{\varepsilon \downarrow 0} \mu(|f_n - f| > \varepsilon) \end{aligned}$$

⇓

$$\lim_{n \rightarrow \infty} |f_n - f|_{\varphi} \leq \lim_{n \rightarrow \infty} (a \cdot \lim_{\varepsilon \downarrow 0} \mu(|f_n - f| > \varepsilon)) \stackrel{\substack{\uparrow \\ \text{regulgl. for} \\ \text{følger m. pos. led}}}{=} a \cdot \lim_{\varepsilon \downarrow 0} (\lim_{n \rightarrow \infty} \mu(|f_n - f| > \varepsilon))$$

$$= a \cdot \lim_{\varepsilon \downarrow 0} 0 = 0$$

↑
 Vi antager utop

$$\lim_{n \rightarrow \infty} \mu(|f_n - f| > \varepsilon) = 0, \varepsilon > 0 \text{ vilk.}$$

□

Påstand:

Antag φ er begrænset og μ er et endeligt mål, da gælder:

$$f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \varphi\text{-mean} \iff f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \mu\text{-mål}$$

Bevis:

Vi benytter den foregående sætn. idet $\mu(\Omega) < \infty \implies$

$$\forall b > 0: \sup_{n \geq 1} \mu(0 < |f_n - f| \leq b) \leq \mu(\Omega) < \infty$$

□

Denne sætn. kan bl.a. benyttes med $\varphi = \arctan$

I tilfældet $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \quad \forall t \in \mathbb{R}_+: \varphi(t) = t^q, \quad 0 < q < \infty$

talr vi i stedet for konvergens i φ -mean om konvergens i q -mean (q -mean = q -middel)

Desuden siger vi, at $f_n \xrightarrow{n \rightarrow \infty} f$ i 0 -mean, hvis $f_n \xrightarrow{n \rightarrow \infty} f$ i μ -mål og $f_n \xrightarrow{n \rightarrow \infty} f$ i ∞ -mean, hvis $f_n \xrightarrow{n \rightarrow \infty} f$ μ -n.o.

Definer

$$\|f\|_0 := \inf \{a \in \overline{\mathbb{R}_+} \mid \mu(|f| > a) \leq a\}$$

$$\|f\|_q := \left(\int_{\Omega} |f|^q d\mu \right)^{1/q}, \quad 0 < q \leq 1$$

$$\|f\|_q := \left\{ \int_{\Omega} |f|^q d\mu \right\}^{1/q}, \quad 1 < q < \infty$$

$$\|f\|_{\infty} := \inf \{a \in \overline{\mathbb{R}_+} \mid \mu(|f| > a) = 0\} = \inf \{c \in \mathbb{R}_+ \mid |f| \leq c \text{ } \mu\text{-n.o.}\}$$

$$M(\mu) := \{f: \Omega \rightarrow \mathbb{R} \mid f \in \overline{M}(\Omega, \mathcal{F})\}$$

Påstand: $\|f-g\|_q$ er en pseudo-metrik på $M(\mu)$, $0 \leq q \leq \infty$

Bevis:

$$\begin{aligned}\|f-g\|_0 &= \inf \{a \in \overline{\mathbb{R}}_+ \mid \mu(|f-g| > a) \leq a\} \\ &= \inf \{a \in \overline{\mathbb{R}}_+ \mid \mu(|g-f| > a) \leq a\} = \|g-f\|_0\end{aligned}$$

$$\|f-g\|_q = \left(\int_{\Omega} |f-g|^q d\mu \right)^{1/q} = \left(\int_{\Omega} |g-f|^q d\mu \right)^{1/q} = \|g-f\|_q, \quad 0 < q \leq 1$$

$$\|f-g\|_q = \left\{ \int_{\Omega} |f-g|^q d\mu \right\}^{1/q} = \left\{ \int_{\Omega} |g-f|^q d\mu \right\}^{1/q} = \|g-f\|_q, \quad 1 \leq q < \infty$$

$$\begin{aligned}\|f-g\|_{\infty} &= \inf \{a \in \overline{\mathbb{R}}_+ \mid \mu(|f-g| > a) = 0\} \\ &= \inf \{a \in \overline{\mathbb{R}}_+ \mid \mu(|g-f| > a) = 0\} = \|g-f\|_{\infty}\end{aligned}$$

Dvs. $\|f-g\|_q = \|g-f\|_q$; $f, g \in M(\mu)$, $q \in \overline{\mathbb{R}}_+$

$$\|f-f\|_0 = \inf \{a \in \overline{\mathbb{R}}_+ \mid \mu(|f-f| > a) \leq a\} = \inf \{a \in \overline{\mathbb{R}}_+ \mid \mu(\emptyset) \leq a\} = 0$$

$$\|f-f\|_q = \left(\int_{\Omega} |f-f|^q d\mu \right)^{1/q} = \left(\int_{\Omega} 0 d\mu \right)^{1/q} = 0, \quad 0 < q \leq 1$$

$$\|f-f\|_q = \left\{ \int_{\Omega} |f-f|^q d\mu \right\}^{1/q} = \left\{ \int_{\Omega} 0 d\mu \right\}^{1/q} = 0, \quad 1 \leq q < \infty$$

$$\|f-f\|_{\infty} = \inf \{a \in \overline{\mathbb{R}}_+ \mid \mu(|f-f| > a) \leq a\} = \inf \{a \in \overline{\mathbb{R}}_+ \mid \mu(\emptyset) \leq a\} = 0$$

Dvs. $\|f-f\|_q = 0$; $f \in M(\mu)$, $q \in \overline{\mathbb{R}}_+$

Lad $a, b \in \mathbb{R}_+$ være vilkårlige ; $f, g \in M(\mu)$

$$\begin{aligned} \{ |f+g| > a+b \} &\subseteq \{ |f|+|g| > a+b \} = \{ |f| > a+b-|g| \} \\ &= \{ \{ |f| > a+b-|g| \} \cap \{ |g| \leq b \} \} \cup \{ \{ |f| > a+b-|g| \} \cap \{ |g| > b \} \} \\ &\subseteq \{ |f| > a \} \cup \{ |g| > b \} \end{aligned}$$

$\Downarrow$ 13.6

$$\mu(|f+g| > a+b) \leq \mu(|f| > a) + \mu(|g| > b) \quad (*)$$

Bm. (*) er også opfyldt for $a, b \in \overline{\mathbb{R}}_+$

$$\|f+g\|_0 := \inf \{ c \in \overline{\mathbb{R}}_+ \mid \mu(|f+g| > c) = 0 \}$$

$$\stackrel{(*)}{\leq} \inf \{ a \in \overline{\mathbb{R}}_+ \mid \mu(|f| > a) = 0 \} + \inf \{ b \in \overline{\mathbb{R}}_+ \mid \mu(|g| > b) = 0 \} = \|f\|_0 + \|g\|_0$$

$$\|f+g\|_q = \left(\int_{\mathbb{R}} |f+g|^q d\mu \right)^{1/q} \stackrel{\uparrow}{\leq} \left(\int_{\mathbb{R}} |f|^q d\mu \right)^{1/q} + \left(\int_{\mathbb{R}} |g|^q d\mu \right)^{1/q} = \|f\|_q + \|g\|_q, \quad 0 < q \leq 1$$

3.12 Minkowski's ulighed

$$\|f+g\|_q = \left\{ \int_{\mathbb{R}} |f+g|^q d\mu \right\}^{1/q} \stackrel{\downarrow}{\leq} \left\{ \int_{\mathbb{R}} |f|^q d\mu \right\}^{1/q} + \left\{ \int_{\mathbb{R}} |g|^q d\mu \right\}^{1/q} = \|f\|_q + \|g\|_q, \quad 1 \leq q < \infty$$

$$\|f+g\|_{\infty} = \inf \{ c \in \overline{\mathbb{R}}_+ \mid \mu(|f+g| > c) = 0 \}$$

$$\stackrel{(*)}{\leq} \inf \{ a \in \overline{\mathbb{R}}_+ \mid \mu(|f| > a) = 0 \} + \inf \{ b \in \overline{\mathbb{R}}_+ \mid \mu(|g| > b) = 0 \} = \|f\|_{\infty} + \|g\|_{\infty}$$

$$\text{Dvs. } \|f+g\|_q \leq \|f\|_q + \|g\|_q ; f, g \in M(\mu), q \in \overline{\mathbb{R}}_+$$

Dermed har vi vist, at $\|f-g\|_q$ er en pseudo-metrik på $M(\mu)$ for $q \in \overline{\mathbb{R}}_+$ jvf. P. 5.

□

Påstand: $\|f\|_0 \leq \varepsilon \implies \mu(|f| > \varepsilon) \leq \|f\|_0 \quad (\varepsilon \in \mathbb{R}_+)$

Beweis:

Antag $\|f\|_0 < \varepsilon$

Bm. $\forall \delta > 0: \varepsilon \in \{a \in \mathbb{R}_+ \mid \mu(|f| > a) \leq a\} \implies \varepsilon + \delta \in \{a \in \mathbb{R}_+ \mid \mu(|f| > a) \leq a\}$

Dvs. $\{a \in \mathbb{R}_+ \mid \mu(|f| > a) \leq a\} = [\|f\|_0; \infty] \vee \{a \in \mathbb{R}_+ \mid \mu(|f| > a) \leq a\} = [\|f\|_0; \infty] \quad (*)$

Hvis $\|f\|_0 = 0$ er påstanden, at $\varepsilon > 0 \implies \mu(|f| > \varepsilon) = 0$

Juf. (*) gælder i dette tilfælde, at $\varepsilon > 0: \mu(|f| > \varepsilon) \leq \varepsilon \quad (+)$

$\varepsilon \mapsto \mu(|f| > \varepsilon)$ er juf. 1.3.2 en aftagende fkt. (og $\lim_{\varepsilon \rightarrow \infty} \mu(|f| > \varepsilon) = 0$), af (+) følger umiddelbart, at $\lim_{\varepsilon \rightarrow 0} \mu(|f| > \varepsilon) = 0$, dette må betyde $\forall \varepsilon > 0: \mu(|f| > \varepsilon) = 0$ (ellers modstrid m. at $\varepsilon \mapsto \mu(|f| > \varepsilon)$ er aft.)

Antag $0 < \|f\|_0 < \varepsilon$ og $\mu(|f| > \varepsilon) > \|f\|_0$

Af (*) følger $0 < \|f\|_0 < \mu(|f| > \varepsilon) \leq \varepsilon$

Idet $\lim_{\varepsilon \rightarrow \infty} \mu(|f| > \varepsilon) = 0 \exists \bar{\varepsilon} > 0 \forall \varepsilon > \bar{\varepsilon}: \mu(|f| > \varepsilon) < \|f\|_0$

$\Downarrow$

Dvs. $\|f\|_0 < \varepsilon \implies \mu(|f| > \varepsilon) \leq \|f\|_0$

Antag $\|f\|_0 = \varepsilon$

Vælg $\{\varepsilon_n\}_{n=1}^\infty \in [\|f\|_0, \infty]$: $\forall n, m \geq 1, n \leq m: \varepsilon_n \geq \varepsilon_m, \lim_{n \rightarrow \infty} \varepsilon_n = \|f\|_0$

Idet $\bigcup_{n=1}^\infty \{|f| > \varepsilon_n\} = \lim_{n \rightarrow \infty} \{|f| > \varepsilon_n\} = \{|f| > \|f\|_0\}$

og $\forall n, m, n \leq m: \{|f| > \varepsilon_n\} \subseteq \{|f| > \varepsilon_m\}$ følger af 1.4.2, at

$\mu(|f| > \|f\|_0) = \lim_{n \rightarrow \infty} \mu(|f| > \varepsilon_n) \leq \|f\|_0$ (idet $\forall n \geq 1: \mu(|f| > \varepsilon_n) \leq \|f\|_0$)

□

Påstand: $\forall \varepsilon \geq 0 : \|f\|_0 \leq \varepsilon + \mu(|f| > \varepsilon)$ ($\forall$ antager $\|f\|_0 < \infty$)

Beweis:

Da $\mu \geq 0$ er påstanden trivial for $\|f\|_0 \leq \varepsilon$

Antag $0 < \varepsilon < \|f\|_0$, dvs. $\|f\|_0 - \varepsilon > 0$

Beweis ved modstrid: Antag $\|f\|_0 > \varepsilon + \mu(|f| > \varepsilon)$

$$\varepsilon \leq \|f\|_0 - \varepsilon < \|f\|_0 : \|f\|_0 - \varepsilon > \mu(|f| > \varepsilon) \geq \mu(|f| > \|f\|_0 - \varepsilon)$$

$\Downarrow$

$\|f\|_0 - \varepsilon \in \{a \in \mathbb{R}_+ \mid \mu(|f| > a) \leq a\}$ hvilket er i modstrid med def. på $\|f\|_0$ $\sum$

$$\|f\|_0 - \varepsilon < \varepsilon < \|f\|_0 : \varepsilon > \|f\|_0 - \varepsilon \geq \mu(|f| > \varepsilon)$$

$\Downarrow$

$\varepsilon \in \{a \in \mathbb{R}_+ \mid \mu(|f| > a) \leq a\}$ hvilket er i modstrid med (*) i det foregående bevis, idet $\varepsilon < \|f\|_0$. $\sum$

Antag $\varepsilon = 0$

Vælg $\{\varepsilon_n\}_{n \geq 1} \in]0, \infty[: \forall n, m \geq 1, n \leq m : \varepsilon_n \geq \varepsilon_m, \lim_{n \rightarrow \infty} \varepsilon_n = 0$

Idet $\bigcap_{n \geq 1} \{|f| > \varepsilon_n\} = \lim_{n \rightarrow \infty} \{|f| > \varepsilon_n\} = \{|f| > 0\}$ og $\forall n, m, n \leq m : \{|f| > \varepsilon_n\} \subseteq \{|f| > \varepsilon_m\}$

følger af 1.4.2, at $\lim_{n \rightarrow \infty} \mu(|f| > \varepsilon_n) = \mu(|f| > 0)$

Da der jvf. det netop viste gælder $\forall n \geq 1 : \|f\|_0 \leq \varepsilon_n + \mu(|f| > \varepsilon_n)$

fås

$$\|f\|_0 \leq \lim_{\varepsilon \rightarrow 0} (\varepsilon + \mu(|f| > \varepsilon)) = \mu(|f| > 0)$$

□

Påstand:

$$\forall 0 \leq q \leq \infty: f_n \xrightarrow{n \rightarrow \infty} f \text{ i } q\text{-middell} \iff \lim_{n \rightarrow \infty} \|f_n - f\|_q = 0$$

Beweis:

$$q=0:$$

' $\implies$ ':

$$\forall \varepsilon > 0: \lim_{n \rightarrow \infty} \mu(|f_n - f| > \varepsilon) = 0$$

$$\text{Velg } \{\varepsilon_n\}_{n \geq 1} \in]0, \infty[: \lim_{n \rightarrow \infty} \varepsilon_n = 0$$

$$\lim_{n \rightarrow \infty} \|f_n - f\|_0 \underset{3.22.19}{\leq} \lim_{n \rightarrow \infty} (\varepsilon_n + \mu(|f_n - f| > \varepsilon_n)) = 0$$

' $\Leftarrow$ ':

$$\lim_{n \rightarrow \infty} \|f_n - f\|_0 = 0$$

Lad $\varepsilon > 0$ være vilkårlig $\exists N \geq 1 \forall n \geq N: \|f_n - f\|_0 \leq \varepsilon$

Dvs. af 3.22.18 følger (jvf. sætn. om følger)

$$\lim_{n \rightarrow \infty} \mu(|f_n - f| > \varepsilon) \overset{3.22.18}{\leq} \lim_{n \rightarrow \infty} \|f_n - f\|_0 = 0$$

$$0 < q \leq 1:$$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f_n - f|^q d\mu = \lim_{n \rightarrow \infty} \|f_n - f\|_q^q \text{ og dermed er påstanden trivial}$$

$$1 \leq q < \infty:$$

' $\Rightarrow$ ':

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f_n - f|^q d\mu = 0 \Rightarrow \lim_{n \rightarrow \infty} \|f_n - f\|_q = \lim_{n \rightarrow \infty} \left\{ \int_{\mathbb{R}} |f_n - f|^q d\mu \right\}^{1/q} = 0$$

$\lim_{x \rightarrow 0} x^{1/q} = 0$
+ requires for sammenhæng af konvergente følger

' $\Leftarrow$ ':

$$\lim_{n \rightarrow \infty} \left\{ \int_{\mathbb{R}} |f_n - f|^q d\mu \right\}^{1/q} = 0 \Rightarrow \lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f_n - f|^q d\mu = 0$$

$x \rightarrow x^{1/q}$
er højere
 $0^{1/q} = 0$

$$q = \infty:$$

' $\Rightarrow$ ':

$$\exists N \in \mathbb{N}(\mu): \lim_{n \rightarrow \infty} (\sup_{\omega \in \mathbb{R}} |f_n(\omega) - f(\omega)|) = 0 \quad (\text{dvs. } f_n \Rightarrow f \text{ } \mu\text{-n.o.})$$

$\Downarrow$

$$\exists N \in \mathbb{N}(\mu) \forall \varepsilon > 0 \exists n' \geq 1 \forall n \geq n' \forall \omega \in \mathbb{R}: |f_n(\omega) - f(\omega)| < \varepsilon$$

$\Downarrow$

$$\forall \varepsilon > 0 \exists n' \geq 1 \forall n \geq n': \mu(|f_n - f| > \varepsilon) = 0$$

$\Downarrow$

$$(\text{def. på } \|\cdot\|_{\infty})$$

$$\forall \varepsilon > 0 \exists n' \geq 1 \forall n \geq n': \|f_n - f\|_{\infty} \leq \varepsilon$$

$\Downarrow$

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{\infty} = 0$$

' $\Leftarrow$ ':

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{\infty} = 0$$

$\Downarrow$

$$\forall \varepsilon > 0 \exists n' \geq 1 \forall n \geq n': \|f_n - f\|_{\infty} < \varepsilon$$

$\Downarrow$ (Bm. ud fra 1.3.2 og def. på $\|f_n - f\|_{\infty}$ fås: $\|f_n - f\|_{\infty} < \varepsilon \Rightarrow \varepsilon \in \{a \in \mathbb{R}_+ \mid \mu(\{f_n - f\} > a) = 0\}$.

$$\forall \varepsilon > 0 \exists n' \geq 1 \forall n \geq n': \mu(\{f_n - f\} > \varepsilon) = 0$$

$\Downarrow$ (foreningsmængd. af tællelig mange nulmængd. er nulmængd. $N_{\varepsilon} = \bigcup_{n=n'}^{\infty} \{f_n - f\} > \varepsilon\}$)

$$\forall \varepsilon > 0 \exists N_{\varepsilon} \in \mathcal{N}(\mu) \exists n' \geq 1 \forall n \geq n', w \in \Omega \setminus N_{\varepsilon}: |f_n(w) - f(w)| < \varepsilon$$

Sæt for en følge $\{\varepsilon_n\}_{n=1}^{\infty} \in]0, \infty[$, $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ $N = \bigcup_{n=1}^{\infty} N_{\varepsilon_n} \in \mathcal{N}(\mu)$ (N_{ε_n} som ovenfor)

da gælder (kan nemt vises ved modstrid):

$$\exists N \in \mathcal{N}(\mu) \forall \varepsilon > 0 \exists n \geq 1 \forall n' \geq n, w \in \Omega \setminus N: |f_{n'}(w) - f(w)| < \varepsilon$$

$\Downarrow$

$$\exists N \in \mathcal{N}(\mu): \lim_{n \rightarrow \infty} \left(\sup_{w \in \Omega \setminus N} |f_n(w) - f(w)| \right) = 0$$

Dvs. $f_n \Rightarrow f$ μ -n.o.

□

Påstand: $\|f\|_q = 0 \iff f = 0 \text{ } \mu\text{-n.o.}, 0 \leq q \leq \infty$

Bevis:

' $\implies$ ':

$q = 0$:

$$\|f\|_0 = 0 \xRightarrow{3.22.18} \forall \varepsilon \geq 0: \mu(|f| > \varepsilon) = 0 \implies f = 0 \text{ } \mu\text{-n.o.}$$

$0 < q < \infty$:

$$\|f\|_q = 0 \implies 0 < q \leq 1: \int_{\mathbb{R}} |f|^q d\mu = 0, \quad 1 \leq q < \infty: \left\{ \int_{\mathbb{R}} |f|^q d\mu \right\}^{1/q} = 0$$

$$\xRightarrow{3.3.8} 0 < q < \infty: f = 0 \text{ } \mu\text{-n.o.}$$

$q = \infty$:

$$\|f\|_{\infty} = 0 \implies \forall \varepsilon > 0: \mu(|f| > \varepsilon) = 0 \text{ (jvf. bm. i følgende bevis)}$$

$$\text{Velg } \{\varepsilon_n\}_{n \in \mathbb{N}} \in]0, \infty[, \varepsilon_n \downarrow 0 \implies \{|f| > \varepsilon_n\} \downarrow \{|f| > 0\}$$

$$\text{Af 1.4.2 følger } \mu(|f| > 0) = 0, \text{ dvs } f = 0 \text{ } \mu\text{-n.o.}$$

' $\impliedby$ ':

$q = 0$:

$$f = 0 \text{ } \mu\text{-n.o.} \implies \mu(|f| > 0) = 0 \implies \|f\|_0 = 0 \text{ (jvf. def. på } \|f\|_0)$$

$0 < q < \infty$:

$$f = 0 \text{ } \mu\text{-n.o.} \xRightarrow{3.3.8} \int_{\mathbb{R}} |f|^q d\mu = 0, \text{ dvs. } 0 < q \leq 1: \|f\|_q = 0$$

$$\implies \left\{ \int_{\mathbb{R}} |f|^q d\mu \right\}^{1/q} = 0, \text{ dvs. } 1 \leq q < \infty: \|f\|_q = 0$$

$q = \infty$:

$$f = 0 \text{ } \mu\text{-n.o.} \implies \mu(|f| > 0) = 0 \implies \|f\|_{\infty} = 0 \text{ (jvf. def. på } \|f\|_{\infty})$$

□

Påstand: $\forall t \geq 0, a \in \mathbb{R} : \|af\|_0 \leq t|a|^{1/2} + \mu(|f| > t|a|^{-1/2})$

Bewis:

Lad $t \geq 0, a \in \mathbb{R}$ være vilkårlig

$$\|af\|_0 \stackrel{3.22.19}{\leq} t|a|^{1/2} + \mu(|af| > t|a|^{1/2}) = t|a|^{1/2} + \mu(|f| > t|a|^{-1/2})$$

□

Påstand: $\forall a \in \mathbb{R}, 0 < q \leq 1 : \|af\|_q = |a|^q \cdot \|f\|_q$

Bewis:

Lad $a \in \mathbb{R}, 0 < q \leq 1$ være vilkårlige

$$\|af\|_q = \int_{\mathbb{R}} |af|^q d\mu \stackrel{3.3.11}{=} |a|^q \int_{\mathbb{R}} |f|^q d\mu = |a|^q \|f\|_q$$

□

Påstand: $\forall a \in \mathbb{R}, 1 \leq q \leq \infty : \|af\|_q = |a| \cdot \|f\|_q$

Bewis:

Lad $a \in \mathbb{R}, 1 \leq q < \infty$ være vilkårlige

$$\|af\|_q = \left\{ \int_{\mathbb{R}} |af|^q d\mu \right\}^{1/q} \stackrel{3.3.11}{=} \left\{ |a|^q \int_{\mathbb{R}} |f|^q d\mu \right\}^{1/q} = |a| \cdot \left\{ \int_{\mathbb{R}} |f|^q d\mu \right\}^{1/q} = |a| \cdot \|f\|_q$$

Lad $a \in \mathbb{R}$ være vilkårlig og $q = \infty$

$$\|af\|_{\infty} = \inf \{k \in \mathbb{R}_+ \mid \mu(|af| > k) = 0\} = a \cdot \inf \{k \in \mathbb{R}_+ \mid \mu(|f| > k) = 0\} = a \cdot \|f\|_{\infty}$$

□

Definition af L^q -rum:

$$\forall 0 \leq q \leq \infty: L^q(\mu) := \{f \in M(\mu) \mid \|f\|_q < \infty\}$$

Påstand:

$$L^0(\mu) \stackrel{(1)}{=} \{f \in M(\mu) \mid \lim_{t \rightarrow \infty} \mu(|f| > t) = 0\}$$

$$\stackrel{(2)}{=} \{f \in M(\mu) \mid \exists t \in \mathbb{R}_+: \mu(|f| > t) < \infty\}$$

$$\stackrel{(3)}{=} M(\mu), \text{ hvis } \mu < \infty$$

Beweis:

$$f \in M(\mu)$$

$$(1), (2) \subseteq:$$

$$\|f\|_0 < \infty, \text{ dvs. } \|f\|_0 \in \mathbb{R}_+ \text{ og } \{a \in \mathbb{R}_+ \mid \mu(|f| > a) < \infty\} = [\|f\|_0, \infty] \quad 3.22.18$$

Lad $\{t_n\}_{n \geq 1} \in \mathbb{R}_+$, $t_n \uparrow \infty, n \rightarrow \infty$ være vilkårlig

$$\exists N \geq 1, t_N \geq \|f\|_0 \text{ og dermed } \mu(|f| > t_N) \leq t_N < \infty \quad (2)$$

$$\text{Def. } \{s_m\}_{m \geq 1} \in \mathbb{R}_+, s_m = t_{N-1+m}, \mu(|f| > s_1) < \infty$$

og jvf. 1.4.3 fås dermed (idet $\lim_{m \rightarrow \infty} s_m = \infty$, $\{|f| > s_m\} \downarrow \emptyset$)

$$\lim_{n \rightarrow \infty} \mu(|f| > t_n) = \lim_{m \rightarrow \infty} \mu(|f| > s_m) = 0, \text{ da } \{t_n\}_{n \geq 1} \text{ var vilk. fås}$$

$$\lim_{t \rightarrow \infty} \mu(|f| > t) = 0 \quad (1)$$

(1), (2) $\geq$:

$$\text{Antag } \lim_{t \rightarrow \infty} \mu(|f| > t) = 0$$

Idet $t \leq s$: $\mu(|f| > s) \stackrel{1.3.2}{\leq} \mu(|f| > t)$, gælder

$$\forall \varepsilon > 0 \exists c \in \mathbb{R}_+ \forall t \geq c: \mu(|f| > t) < \varepsilon$$

$$\forall t > c \wedge \varepsilon: \mu(|f| > t) \leq \varepsilon, \text{ dvs } t \in \{a \in \mathbb{R}_+ \mid \mu(|f| > a) \leq a\}$$

$$\Rightarrow \|f\|_0 = \inf \{a \in \mathbb{R}_+ \mid \mu(|f| > a) \leq a\} \leq t < \infty \quad (1)$$

$$\text{Antag } \exists t \in \mathbb{R}_+: \mu(|f| > t) < \infty$$

$$\text{Dvs. } \exists t \in \mathbb{R}_+, c \in \mathbb{R}_+: \mu(|f| > t) = c$$

$$\forall \delta > 0: \mu(|f| > t) \stackrel{1.3.2}{\geq} \mu(|f| > t + \delta), \text{ dvs } \mu(|f| > c + t) \leq c \leq c + t$$

$$c + t \in \{a \in \mathbb{R}_+ \mid \mu(|f| > a) \leq a\} \Rightarrow \|f\|_0 \leq c + t < \infty$$

(3)

Hvis $\mu < \infty$ gælder specielt $\forall t \in \mathbb{R}_+: \mu(|f| > t) < \infty$
for vilk. $f \in M(\mu)$. Derved følger det enskede af (2)

□

Påstand: $\forall 0 < q < \infty: L^q(\mu) = \{f \in M(\mu) \mid \int_{\mathbb{R}} |f|^q d\mu < \infty\}$

Bevis:

$\subseteq$:

$$\|f\|_q < \infty$$

$$\text{Dvs. } \int_{\mathbb{R}} |f|^q d\mu < \infty, \quad 0 < q \leq 1 \quad \text{og} \quad \left\{ \int_{\mathbb{R}} |f|^q d\mu \right\}^{1/q} < \infty, \quad 1 \leq q < \infty$$

$\Downarrow$

$$\int_{\mathbb{R}} |f|^q d\mu < \infty, \quad 0 < q < \infty$$

$\supseteq$:

$$\int_{\mathbb{R}} |f|^q d\mu < \infty \Rightarrow \int_{\mathbb{R}} |f|^q d\mu < \infty, \quad 0 < q \leq 1 \quad \text{og} \quad \left\{ \int_{\mathbb{R}} |f|^q d\mu \right\}^{1/q}, \quad 1 \leq q < \infty$$

□

Påstand: $L^\infty(\mu) = \{f \in M(\mu) \mid \exists c \in \mathbb{R}_+: \mu(|f| > c) = 0\}$

Bevis:

$$\|f\|_\infty < \infty \iff \exists c \in \mathbb{R}_+: c \in \{a \in \mathbb{R}_+ \mid \mu(|f| > a) = 0\}$$

$$\iff \exists c \in \mathbb{R}_+: \mu(|f| > c) = 0$$

□

Påstand: $\forall q \in \bar{\mathbb{R}}_+ : (L^q(\mu), \|\cdot\|_q)$ er et pseudo-metrisk vektorrum

Beweis:

Da $\forall q \in \bar{\mathbb{R}}_+ : L^q(\mu) \subseteq M(\mu)$ og da vi allerede har vist, at $\|\cdot\|_q$ er en pseudo-metrik på $M(\mu)$ skal vi blot vise, at $\forall q \in \bar{\mathbb{R}}_+ : L^q(\mu)$ er et vektorrum.

Lad $q \in \bar{\mathbb{R}}_+$ være vilk., hvis $\|f\|_q, \|g\|_q < \infty$ følger, da $\|\cdot\|_q$ er en pseudo-metrik på $L^q(\mu)$, at $\|f+g\|_q \leq \|f\|_q + \|g\|_q < \infty$
Dvs. $f, g \in L^q(\mu) \Rightarrow f+g \in L^q(\mu)$

Lad $a \in \mathbb{R}$ være vilkårlig.

Antag $f \in L^q(\mu)$, dvs. $\|f\|_q < \infty$

$q=0$:

$$\|a \cdot f\|_0 \stackrel{3.22.22}{=} \|f\|_0 |a| + \mu(\{f \neq 0\}) \stackrel{3.22.18}{\leq} \|f\|_0 |a| + \|f\|_0 < \infty$$

$t = \|f\|_0 |a|^{1/2}$

$0 < q \leq 1$:

$$\|a \cdot f\|_q = |a|^q \|f\|_q < \infty$$

$1 \leq q \leq \infty$:

$$\|a \cdot f\|_q = |a| \cdot \|f\|_q < \infty$$

Dvs. $\|a \cdot f\|_q < \infty$ og dermed $a \cdot f \in L^q(\mu)$

□

Påstand: addition og skalarmultiplikation er kontinuerlige
funktioner på det pseudo-metriske vektorrum $(L^q(\mu), \|\cdot\|_q)$
(hh. $(f, g) \rightarrow f+g, (a, f) \rightarrow a \cdot f$)

Bevis:

Lad $\varepsilon > 0$ være vilkårlig og $f_0, g_0 \in L^q(\mu)$ være givet

Vurdering:

$$\|f_0 + g_0 - f - g\|_q = \|f_0 - f + g_0 - g\|_q \leq \|f_0 - f\|_q + \|g_0 - g\|_q$$

Hvis $\|f_0 - f\|_q < \delta$, $\|g_0 - g\|_q < \delta$, hvor $\delta = \varepsilon/2$ gælder således

$$f_0, g_0 \in L^q(\mu) \quad \forall \varepsilon > 0 \exists \delta > 0 \forall f, g: \|f_0 - f\|_q, \|g_0 - g\|_q < \delta \Rightarrow \|f_0 + g_0 - f - g\|_q < \varepsilon$$

Derfor addition er en kont. fkt. på $L^q(\mu)$.

Lad $\varepsilon > 0$ være vilkårlig og $a \in \mathbb{R}, f_0 \in L^q(\mu)$ være givet

Resten vises tilsvarende i.h.t. 3.22.22-24

□

Påstand: $\lim_{n \rightarrow \infty} \|f_n - g\|_q = 0$, $\lim_{n \rightarrow \infty} \|f_n - h\|_q = 0 \implies g = h$ μ -u.o. $q \in \overline{\mathbb{R}}_+$

Beweis: Folger af uligheden: $\forall q \in \overline{\mathbb{R}}_+ : \|f + g\|_q \leq \|f\|_q + \|g\|_q \quad \square$

$$V \subseteq L^1(\mu) \quad ; \quad v: \mathcal{F} \rightarrow \overline{\mathbb{R}}_+, \quad \forall F \in \mathcal{F}: v(F) = \sup \left\{ \int_F |f| d\mu \mid f \in V \right\}$$

V er uniformt integrabel, hvis

$$v(\Omega) < \infty, \quad \forall \{F_n\}_{n=1}^\infty \subseteq \mathcal{F}, \quad F_n \downarrow \emptyset, \quad \lim_{n \rightarrow \infty} v(F_n) = 0$$

V er endeligt μ -koncentreret, hvis

$$v(\Omega) < \infty, \quad \forall \varepsilon > 0 \exists A \in \mathcal{F}: v(A^c) < \varepsilon \wedge \mu(A) < \infty$$

Påstand: V er σ -subadditiv og μ -satureret

Bevis:

$$\{F_n\}_{n=1}^\infty \subseteq \mathcal{F}, \quad F \in \mathcal{F}: \quad F \subseteq \bigcup_{n=1}^\infty F_n$$

$$v(F) = \sup \left\{ \int_F |f| d\mu \mid f \in V \right\} \stackrel{\substack{3.3.3 \\ F \subseteq \bigcup_{n=1}^\infty F_n}}{\leq} \sup \left\{ \int_\Omega \left(\sum_{n=1}^\infty 1_{F_n} \right) |f| d\mu \mid f \in V \right\}$$

$$\stackrel{3.7}{=} \sup \left\{ \sum_{n=1}^\infty \int_\Omega 1_{F_n} |f| d\mu \mid f \in V \right\} \leq \sum_{n=1}^\infty \sup \left\{ \int_\Omega 1_{F_n} |f| d\mu \mid f \in V \right\} = \sum_{n=1}^\infty v(F_n)$$

Dvs. V er σ -subadditiv

Bm. jvf. def. på integralitet fås $N \in \mathcal{N}(\mu) \Rightarrow v(N) = 0$

Dvs. v er μ -satureret

□

Påstand: $\mu < \infty \implies V$ er endeligt μ -koncentreret hviss. $V(\Omega) < \infty$

Bewis:

Antag $\mu < \infty$

Lad $\varepsilon > 0$ være vilkårlig

$V(\emptyset) = 0 < \varepsilon$, $\mu(\Omega) < \infty$ hvorfra vi ser, at V er endeligt μ -koncentreret hviss. $V(\Omega) < \infty$

□

V siges at være L^1 -domineret hviss $\exists h \in L^1(\mu) \forall f \in V: |f| \leq h$ μ -n.o.

Påstand: V er L^1 -domineret

$\implies V$ er uniformt integrabel og endeligt μ -koncentreret

(Bm. specielt er enhver endelig delmængde af L^1 unif. int. og end. μ -konc.)

Bewis:

V er L^1 -domineret, dvs $\exists h \in L^1(\mu) \forall f \in V: |f| \leq h$ μ -n.o.

Lad $F \in \mathcal{F}$ være vilkårlig:

$$V(F) = \sup \left\{ \int_F |f| d\mu \mid f \in V \right\} \leq \int_F h d\mu \quad \text{3.3.3}$$

$$\forall n \geq 1 \quad A_n := \{ |h| > \frac{1}{n} \}$$

$$\lim_{n \rightarrow \infty} V(A_n^c) \leq \lim_{n \rightarrow \infty} \int_{A_n^c} |h| d\mu = \int_{\{ |h| = 0 \}} |h| d\mu = \int 0 d\mu = 0 \quad \text{3.13.1}$$

$$\text{Dvs. } \forall \varepsilon > 0 \exists n \geq 1: V(A_n^c) < \varepsilon$$

$$\mu(A_n)/n \leq \int_{A_n} |h| d\mu < \infty \implies \mu(A_n) < \infty \quad \begin{matrix} \text{3.9} \\ \text{+1.3.2} \end{matrix}$$

$$\text{Dvs. } \forall \varepsilon > 0 \exists A \in \mathcal{F}: V(A^c) < \infty \wedge \mu(A) < \infty$$

□

ν siges at være uniformt begrænset, hvis $\exists a \in \mathbb{R}_+ \forall f \in \mathcal{V}: |f| \leq a$ μ -n.o.

Påstand: $\exists a \in \mathbb{R}_+ \forall f \in \mathcal{V}: |f| \leq a$ μ -n.o. (dvs. ν er uniformt begrænset)

$\Downarrow$

$$\forall f \in \mathcal{V}; F, A \in \mathcal{F}: \int_F |f| d\mu \leq a \cdot \mu(F \cap A) + \nu(A^c) \quad (*)$$

Bevis:

$\exists a \in \mathbb{R}_+ \forall f \in \mathcal{V}: |f| \leq a$ μ -n.o., lad $F, A \in \mathcal{F}$ være vilkårlig

$\Downarrow$

$$\int_F |f| d\mu \leq \int_{F \cap A} a d\mu + \int_{A^c} |f| d\mu = a \cdot \mu(F \cap A) + \int_{A^c} |f| d\mu \leq a \cdot \mu(F \cap A) + \nu(A^c)$$

$\begin{matrix} \uparrow & & \uparrow & & \uparrow \\ \text{3.3.3} & & \text{3.3.9} & & \text{def. p. 2} \\ \text{3.3.12} & & \text{3.8.1} & & \nu(A^c) \end{matrix}$

□

Påstand: $\exists a \in \mathbb{R}_+ \forall f \in \mathcal{V}: |f| \leq a$ μ -n.o.

$$\wedge \nu(\Omega) < \infty \wedge \forall \varepsilon > 0 \exists A \in \mathcal{F}: (\nu(A^c) < \varepsilon \wedge \mu(A) < \infty)$$

$\Downarrow$

$$\nu(\Omega) < \infty, \forall \{F_n\}_{n \geq 1} \subseteq \mathcal{F}, F_n \downarrow \emptyset: \lim_{n \rightarrow \infty} \nu(F_n) = 0$$

Bevis:

Af (*) følger $\forall f \in \mathcal{V}; F, A \in \mathcal{F}: \int_F |f| d\mu \leq a \cdot \mu(F \cap A) + \nu(A^c)$

Juf. antagelse $\exists \{A_n\}_{n \geq 1} \subseteq \mathcal{F} \forall \varepsilon > 0 \exists N \geq 1 \forall n \geq N: \nu(A_n^c) < \varepsilon/2$

Lad $\{F_n\}_{n \geq 1} \subseteq \mathcal{F}, F_n \downarrow \emptyset$ være vilkårlig af 1.3.2 og 1.43 følger

$$\forall \varepsilon > 0 \exists N_2 \geq 1 \forall n \geq N_2: \mu(F_n \cap A_n) \leq \mu(F_n) < \frac{\varepsilon}{a^2} \Rightarrow a \cdot \mu(F_n \cap A_n) < \varepsilon/2$$

Lad $\varepsilon > 0$ være givet, vælg N_1, N_2 som ovenfor sæt $N = N_1 \vee N_2$

$$\forall n \geq N: a \mu(F_n \cap A_n) + \nu(A_n^c) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Af antagelse følger dermed: $\nu(\Omega) < \infty \wedge \forall \{F_n\}_{n \geq 1} \subseteq \mathcal{F}, F_n \downarrow \emptyset: \lim_{n \rightarrow \infty} \nu(F_n) = 0$

□

Påstand:

Lad $0 < c < \infty$ være givet

Antag $V, W \in L^1(\mu)$ er uniformt integrable

Da er $\mathcal{U}_c := \{f \in M(\Omega, \mathbb{R}) \mid \exists g \in V, h \in W: |f| \leq c(|g| + |h|) \mu\text{-n.o.}\}$ unif. int.

Bevis:

$$V_{\mathcal{U}_c}(\Omega) \leq c \cdot (V_V(\Omega) + V_W(\Omega)) < \infty$$

$\uparrow$
(j.f. def. på V_V og $\mathcal{U}_c$)

Lad $\{F_n\}_{n \geq 1} \subseteq \mathcal{F}$, $F_n \downarrow \emptyset$, $\varepsilon > 0$ være vilkårlig

Af antagelserne fås $\exists N \geq 1 \forall n \geq N: V_V(F_n) < \frac{\varepsilon}{c \cdot 2}$, $V_W(F_n) < \frac{\varepsilon}{c \cdot 2}$

Af def. på $\mathcal{U}_c$ ses $V_{\mathcal{U}_c}(F_n) \leq c(V_V(F_n) + V_W(F_n)) < \varepsilon$

□

3.23 Dunford-Pettis' Sætning

$$(\Omega, \mathcal{F}, \mu) \text{ m}, \{f_n\}_{n \geq 1} \subseteq M(\Omega, \mathbb{R})$$

$$f^* := \limsup_{n \rightarrow \infty} f_n, \quad f_* := \liminf_{n \rightarrow \infty} f_n$$

$$g, h \in M(\Omega, \mathbb{R}): f^* \leq g, \quad h \leq f_*$$

(1)

Hvis $\{f_n^+ | n \geq 1\}$ er uniformt integrabel gælder, at $(f_*)^+ \in L^1(\mu)$

og hvis $F \in \mathcal{F}: 1_F g \in L^1(\mu)$ gælder: $\limsup_{n \rightarrow \infty} \int_F f_n d\mu \leq \int_F g d\mu$

(2)

Hvis $\{f_n^- | n \geq 1\}$ er uniformt integrabel gælder, at $(f^*)^- \in L^1(\mu)$

og hvis $F \in \mathcal{F}: 1_F h \in L^1(\mu)$ gælder: $\int_F h d\mu \leq \liminf_{n \rightarrow \infty} \int_F f_n d\mu$

(3)

Hvis $\{f_n | n \geq 1\}$ er uniformt integrabel gælder, at $(f_*)^+, (f^*)^- \in L^1(\mu)$

$\forall F \in \mathcal{F}$:

$$\int_F f_* d\mu \leq \liminf_{n \rightarrow \infty} \int_F f_n d\mu \leq \limsup_{n \rightarrow \infty} \int_F f_n d\mu \leq \int_F f^* d\mu$$

Hvis $f_n \xrightarrow{n \rightarrow \infty} f$ μ -n.o., da gælder $\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$ og $\int_F f_n d\mu \xrightarrow{n \rightarrow \infty} \int_F f d\mu$
(boks 3.22.20)

Beweis:

(1)

$V := \{f_n^+ | n \geq 1\}$ er uniformt integrabel

$$(f_*)^+ = (\sup_{j \geq 1} \inf_{n \geq j} f_n)^+ = \sup_{j \geq 1} (\inf_{n \geq j} f_n)^+ = \sup_{j \geq 1} \inf_{n \geq j} f_n^+ = \liminf_{n \rightarrow \infty} f_n^+$$

$$\text{Juf. P.4.21: } f_* \leq f^* \leq g \implies g^- \leq (f_*)^- = \liminf_{n \rightarrow \infty} f_n^- \quad (*)$$

$$\limsup_{n \rightarrow \infty} f_n^+ = (f^*)^+ \leq g^+$$

$$\begin{aligned} \int_{\Omega} (f_*)^+ d\mu &\stackrel{(*)}{=} \int_{\Omega} \liminf_{n \rightarrow \infty} f_n^+ d\mu \stackrel{3.5}{\leq} \liminf_{n \rightarrow \infty} \int_{\Omega} f_n^+ d\mu \\ &= \sup_{j \geq 1} \inf_{n \geq j} \int_{\Omega} f_n^+ d\mu \leq \sup_{n \geq 1} \int_{\Omega} f_n^+ d\mu = V(\Omega) < \infty \\ &\quad \uparrow \\ &\quad V \text{ unif. int.} \end{aligned}$$

$$\text{Dvs. } (f_*)^+ \in L^1(\mu)$$

Lad $\varepsilon > 0$ være vilkårlig

$$\forall n \geq 1: g_n := (f_n^+ - g^+)^+, \quad (\infty - \infty = 0)$$

$$0 \leq g_n \leq f_n^+ \leq g_n + g^+, \quad g^+ \geq (f^*)^+ = \limsup_{n \rightarrow \infty} f_n^+ \implies \lim_{n \rightarrow \infty} g_n = 0 \quad (†)$$

$j \geq 1$:

$$A_j := \bigcup_{n=j}^{\infty} \{g_n \geq 1\}, \quad B_j := \bigcap_{n=1}^j \{g_n \leq 2^{-j}\}, \quad B := \bigcup_{n=1}^{\infty} \{g_n > 0\}$$

$$B_j \cap B \downarrow \emptyset, \quad \text{og } (†) \implies A_j \downarrow \emptyset$$

Da V er uniformt integrabel $\exists p \geq 1 \forall n \geq p: V(B_n \cap B) < \varepsilon, V(A_n) < \varepsilon$

$$\text{Lad } n \geq p: C_n := (A_n \cup B_n)^c = A_n^c \cap B_n^c$$

$$\mu(C_n) \stackrel{1.3.2}{\leq} \mu(B_n^c) = \mu\left(\bigcup_{i=1}^n \{g_i > 2^{-n}\}\right) \leq \sum_{i=1}^n \mu(g_i > 2^{-n})$$

$$\stackrel{3.9}{\leq} \sum_{i=1}^n 2^n \cdot \int_{\mathbb{R}} g_i d\mu \stackrel{(4)}{\leq} \sum_{i=1}^n 2^n \cdot \int_{\mathbb{R}} f_i^+ d\mu \leq n \cdot 2^n \cdot v(\mathbb{R}) < \infty$$

$$C_p \subseteq A_p^c = \bigcap_{i=p}^{\infty} \{g_i < 1\} \Rightarrow \forall n \geq p: 0 \leq 1_{C_p} \cdot g_n \leq 1_{C_p}$$

$$\int_{\mathbb{R}} 1_{C_p} d\mu \stackrel{3.8.1}{=} \mu(C_p) < \infty, \text{ dvs. } 1_{C_p} \in L^1(\mu)$$

Juf. (4) gælder $\lim_{n \rightarrow \infty} 1_{C_p} \cdot g_{n+p-1} = 0$ da $\{1_{C_p} g_n\}_{n \geq p}$ juf. det ovenstående er domineret af $1_{C_p} \in L^1(\mu)$, følger af Lebesgue's sætning om domineret konvergens, at

$$\exists q \geq p \forall n \geq q: \left| \int_{\mathbb{R}} 1_{C_p} \cdot g_n d\mu \right| = \int_{\mathbb{R}} 1_{C_p} \cdot g_n d\mu < \varepsilon$$

Lad $n \geq q$:

$$\int_{\mathbb{R}} g_n d\mu \stackrel{3.3.3}{\leq} \int_{A_p} g_n d\mu + \int_{B_p} g_n d\mu + \int_{C_p} g_n d\mu$$

$$\stackrel{3.3.4}{=} \int_{A_p} g_n d\mu + \int_{B_p \cap B} g_n d\mu + \int_{C_p} g_n d\mu$$

$$\stackrel{(4)}{<} \int_{A_p} f_n d\mu + \int_{B_p \cap B} f_n d\mu + \varepsilon \stackrel{3.3.3}{\leq} v(A_p) + v(B_p \cap B) + \varepsilon < 3\varepsilon$$

& ovenstående

Lad $F \in \mathcal{F}$, antag $1_F \cdot g \in L^1(\mu)$

$$\int_F g d\mu = \infty:$$

$V \subseteq L^1(\mu)$ (pr. def. af uniform integrabilitet), dette medfører
 $\forall n \geq 1: \int_F f_n^+ d\mu < \infty$, og pr. def. af μ -integrabilitet dermed $V \subseteq L^1(\mu)$

Heraf følger, at $\limsup_{n \rightarrow \infty} \int_F f_n d\mu$ (og $\liminf_{n \rightarrow \infty} \int_F f_n d\mu$) er veldefineret.

$$\text{Helt trivielt gælder: } \limsup_{n \rightarrow \infty} \int_F f_n d\mu \leq \infty$$

$$\int_F g d\mu < \infty:$$

$$\limsup_{n \rightarrow \infty} \int_F f_n d\mu = \limsup_{n \rightarrow \infty} \left\{ \int_F f_n^+ d\mu - \int_F f_n^- d\mu \right\}$$

$$\stackrel{\text{P.4.11}}{\leq} \limsup_{n \rightarrow \infty} \int_F f_n^+ d\mu + \limsup_{n \rightarrow \infty} \int_F -f_n^- d\mu$$

$$\stackrel{\text{P.4.20}}{=} \limsup_{n \rightarrow \infty} \int_F f_n^+ d\mu - \liminf_{n \rightarrow \infty} \int_F f_n^- d\mu$$

$$\stackrel{\text{3.3.3}}{\stackrel{(+)}{\leq}} \limsup_{n \rightarrow \infty} \int_F (g_n + g^+) d\mu - \liminf_{n \rightarrow \infty} \int_F f_n^- d\mu$$

$$\stackrel{\text{3.5}}{\leq} \limsup_{n \rightarrow \infty} \int_F (g_n + g^+) d\mu - \int_F \liminf_{n \rightarrow \infty} f_n^- d\mu$$

$$\stackrel{\text{3.3.3}}{\stackrel{*}{\leq}} \limsup_{n \rightarrow \infty} \int_F (g_n + g^+) d\mu - \int_F g^- d\mu$$

$$\stackrel{\text{3.3.12}}{\leq} \limsup_{n \rightarrow \infty} \left\{ \int_F g_n d\mu + \int_F g^+ d\mu \right\} - \int_F g^- d\mu$$

$$\leq \limsup_{n \rightarrow \infty} \{3\varepsilon + \int_F g^+ d\mu\} - \int_F g^- d\mu$$

$$= 3\varepsilon + \int_F g^+ d\mu - \int_F g^- d\mu = 3\varepsilon + \int_F g d\mu$$

$\varepsilon > 0$ var vilkårlig, og dermed har vi vist det ønskede

□

(2):

Antag $\mathcal{V} := \{f_n^- | n \geq 1\} = \{(-f_n)^+ | n \geq 1\}$ er uniformt integrabel

Af (1) følger, at $(\liminf_{n \rightarrow \infty} (-f_n))^+ \stackrel{\text{P. 4.20}}{=} (-\limsup_{n \rightarrow \infty} f_n)^+ = (\limsup_{n \rightarrow \infty} f_n)^- = (f^*)^- \in L^1(\mu)$

$$\limsup_{n \rightarrow \infty} (-f_n) \stackrel{\text{P. 4.20}}{=} -\liminf_{n \rightarrow \infty} f_n \leq -h$$

Hvis $F \in \mathcal{F}$: $1_F \cdot h \in L(\mu)$ følger dermed af (1), at

$$\limsup_{n \rightarrow \infty} \int_F (-f_n) d\mu \leq \int_F (-h) d\mu$$

$\Downarrow$

$$-\limsup_{n \rightarrow \infty} \int_F (-f_n) d\mu \stackrel{\text{P. 4.20}}{=} \liminf_{n \rightarrow \infty} (-\int_F (-f_n) d\mu) \stackrel{\text{3.3.11}}{=} \liminf_{n \rightarrow \infty} \int_F f_n d\mu \geq \int_F h d\mu \stackrel{\text{3.3.11}}{=} -\int_F -h d\mu$$

□

(3):

Antag $\mathcal{V} := \{f_n | n \geq 1\}$ er uniformt integrabel

Af definitionerne på uniform integrabilitet og integral ser vi, at dette medfører, at $\mathcal{V}^- := \{f_n^- | n \geq 1\}$, $\mathcal{V}^+ := \{f_n^+ | n \geq 1\}$ er unif. int.

Af (1) og (2) følger heraf, at $(f_*)^+, (f^*)^- \in L^1(\mu) \Rightarrow f_*, f^* \in L(\mu)$

$$\Rightarrow \forall F \in \mathcal{F}: 1_F \cdot f_*, 1_F \cdot f^* \in L(\mu)$$

Sættes vi $f_* = h$, $f^* = g$ i hh. (1) og (2) fås $\forall F \in \mathcal{F}$:

$$\int_F \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_F f_n d\mu \leq \limsup_{n \rightarrow \infty} \int_F f_n d\mu \leq \int_F \limsup_{n \rightarrow \infty} f_n d\mu$$

Antag yderligere, at $f_n \xrightarrow{n \rightarrow \infty} f$ μ -n.o.

$\Downarrow$

$$f_* = f = f^* \text{ } \mu\text{-n.o.}$$

$\Downarrow$

$$(f_*)^+ = f^+ \text{ } \mu\text{-n.o.} \wedge (f^*)^- = f^- \text{ } \mu\text{-n.o.}, \text{ da } (f_*)^+, (f^*)^- \in L^1(\mu)$$

følger jvf. 3.3.4, og def. på integralit, at $f \in L^1(\mu)$

Af 3.22.37 følger, at $\mathcal{N} := \{f\}$ er uniformt integrabel

3.22.39 giver efter følgende at $\{f_n - f | n \geq 1\}$ er uniformt integrabel, da $|f_n - f| \leq |f_n| + |f|$.

Af ulighederne

$$\int_F \liminf_{n \rightarrow \infty} |f_n - f| d\mu \leq \liminf_{n \rightarrow \infty} \int_F |f_n - f| d\mu \leq \limsup_{n \rightarrow \infty} \int_F |f_n - f| d\mu \leq \int_F \limsup_{n \rightarrow \infty} |f_n - f| d\mu, \quad F \in \mathcal{F}$$

følger, idet $f_* = f = f^* \text{ } \mu\text{-n.o.} \Rightarrow \liminf_{n \rightarrow \infty} |f_n - f| = \limsup_{n \rightarrow \infty} |f_n - f| = 0 \text{ } \mu\text{-n.o.}$, at

$$0 \stackrel{3.3.4}{=} \int_F \liminf_{n \rightarrow \infty} |f_n - f| d\mu \leq \liminf_{n \rightarrow \infty} \int_F |f_n - f| d\mu \leq \limsup_{n \rightarrow \infty} \int_F |f_n - f| d\mu \leq \int_F \limsup_{n \rightarrow \infty} |f_n - f| d\mu \stackrel{3.3.4}{=} 0, \quad F \in \mathcal{F}$$

$\Downarrow$

$$\liminf_{n \rightarrow \infty} \int_F |f_n - f| d\mu = \limsup_{n \rightarrow \infty} \int_F |f_n - f| d\mu = 0, \quad F \in \mathcal{F}$$

$\Downarrow$

$$\lim_{n \rightarrow \infty} \int_F |f_n - f| d\mu = 0, \quad F \in \mathcal{F} \quad \text{specielt } \lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f_n - f| d\mu = 0 \text{ dvs. } f_n \rightarrow f \text{ i 1-norm}$$

Af $f_* = f = f^* \text{ } \mu\text{-n.o.}$ følger desuden

$$\int_F f d\mu \stackrel{3.3.3}{=} \int_F f_* d\mu \leq \liminf_{n \rightarrow \infty} \int_F f_n d\mu \leq \limsup_{n \rightarrow \infty} \int_F f_n d\mu \leq \int_F f^* d\mu \stackrel{3.3.3}{=} \int_F f d\mu$$

$$\text{og dermed } \lim_{n \rightarrow \infty} \int_F f_n d\mu = \int_F f d\mu$$

□

3.24 Kriterier for uniform integrabilitet

$(\Omega, \mathcal{F}, \mu) \in \mathcal{M}$, $V \subseteq L^1(\mu)$ (i det flg. def. $v: \mathcal{F} \rightarrow \mathbb{R}_+$ som i 3.22)

Følgende udsagn er ækvivalente

(1) V er uniformt integrabel

(2) V er μ -kontinueret og V er endeligt μ -koncentreret

(3) V er endeligt μ -koncentreret

$\forall \varepsilon > 0 \exists \rho: \mathcal{F} \rightarrow \mathbb{R}_+$, ρ μ -kont., $S \subseteq V$, S endelig:

$$\forall f \in V \setminus S, F \in \sigma(\mathcal{F}): \int_F |f| d\mu < \varepsilon + \rho(F)$$

(4) V er endeligt μ -koncentreret

$$\forall \varepsilon > 0 \exists a \in \mathbb{R}_+ \forall f \in V: \int_{\{|f| > a\}} |f| d\mu < \varepsilon$$

(5) V er endeligt μ -koncentreret

$\exists c \in \mathbb{R}_+$, $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ bounded.

$$\forall f \in V: \lim_{t \rightarrow \infty} \frac{\varphi(t)}{t} = \infty \wedge \int_{\Omega} \varphi(|f|) d\mu \leq c$$

Bevis:

(1) $\Rightarrow$ (2):

Dvs. vi antager, at ν er uniformt integrabel.

Af 3.22.36 følger, at ν er σ -subadditiv og μ -satureret
hvoraf vi ud fra 3.16.6 kan slutte, at ν er μ -kontinueret.

Desuden gælder pr. def. af uniform integrabilitet, at $\nu(\Omega) < \infty$
og $\exists f_1 \in V: \nu(\Omega) \leq 2^{-1} + \int_{\Omega} |f_1| d\mu$

$\{0 < |f_1| < 2^{-1}\} \downarrow \emptyset, j \uparrow \infty \Rightarrow \exists a_1 > 0: \nu(0 < |f_1| < a_1) < 2^{-1}$

$F_1 := \{|f_1| \geq a_1\}, \exists f_2 \in V: \nu(\Omega \setminus F_1) \leq 2^{-2} + \int_{\Omega \setminus F_1} |f_2| d\mu$

Vi ser $\exists \{a_n\}_{n \geq 1} \in \mathbb{R}_+ \setminus \{0\}, \{f_n\}_{n \geq 1} \in V$ således

$\forall n \geq 0: \nu(\Omega \setminus F_n) \leq 2^{-n-1} + \int_{\Omega \setminus F_n} |f_{n+1}| d\mu, \nu(H_n) < 2^{-n-1}$

hvor $F_0 = \emptyset, n \geq 1: F_n = \bigcup_{j=1}^n \{|f_j| \leq a_j\}, H_{n-1} := \{0 < |f_n| < a^n\}, G_{n-1} := \{f_n = 0\}$

Sæt $F = \bigcup_{n \in \mathbb{N}} F_n \Rightarrow F \setminus F_n \downarrow \emptyset \xrightarrow[\text{unif. int. af } \nu]{} \Rightarrow \nu(F \setminus F_n) \xrightarrow{n \rightarrow \infty} 0$

$$\begin{aligned} \nu(\Omega \setminus F_n) &\leq 2^{-n-1} + \int_{\Omega \setminus F_n} |f_{n+1}| d\mu \\ &\stackrel{\Omega \setminus F_n \subseteq H_n \cup F \setminus F_n \cup G_n}{\leq 2^{-n-1}} + \int_{H_n} |f_{n+1}| d\mu + \int_{F \setminus F_n} |f_{n+1}| d\mu + \int_{G_n} |f_{n+1}| d\mu \\ &\stackrel{3.3.3+12}{\leq 2^{-n-1}} + \nu(H_n) + \nu(F \setminus F_n) < 2^{-n-1} + 2^{-n-1} + \nu(F \setminus F_n) \\ &\stackrel{\int_{G_n} |f_{n+1}| d\mu = 0}{\stackrel{\text{+ def. pr } \nu}} = 2^{-n} + \nu(F \setminus F_n) \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Dvs. $\nu(\Omega \setminus F_n) \xrightarrow{n \rightarrow \infty} 0$

$$\mu(F_n) \leq \sum_{j=1}^n \mu(|f_j| \geq a_j) \stackrel{\substack{3.9 \\ \text{Markov}}}{\leq} \sum_{j=1}^n \frac{1}{a_j} \int_{\mathcal{R}} |f_j| d\mu \leq \nu(\mathcal{R}) \sum_{j=1}^n < \infty, \quad n \geq 1$$

$$\mu(F_0) = 0 < \infty$$

Af def. 3.22.35 ses vi, at ν er endeligt μ -koncentreret $\square$

$$(2) \Rightarrow (3)$$

At ν er endeligt μ -koncentreret er antaget

Lad $\varepsilon > 0$ være vilkårlig

Set $\rho = \nu$, $\nu: \mathcal{F} \rightarrow \overline{\mathbb{R}}_+$, ν er pr. antagelse μ -kontinueret

$$\forall f \in \mathcal{V}, F \in \sigma(f) \subseteq \mathcal{F}: \int_F |f| d\mu \leq \nu(F) < \nu(F) + \varepsilon$$

$\square$

$$(3) \Rightarrow (4)$$

At ν er endeligt μ -koncentreret er antaget

Lad $\varepsilon > 0$ være vilkårlig, pr. antagelse gælder:

$\exists \rho: \mathcal{F} \rightarrow \overline{\mathbb{R}}_+$, ρ er μ -kont., $S \subseteq \mathcal{V}$, S er endelig:

$$\forall f \in \mathcal{V} \setminus S, F \in \sigma(f): \int_F |f| d\mu < \varepsilon/2 + \rho(F) \quad (+)$$

$$\text{Da } \rho \text{ er } \mu\text{-kontinueret } \exists \delta > 0 \forall F \in \mathcal{F}, \mu(F) \leq \delta: \rho(F) \leq \varepsilon/2 \quad (*)$$

Da ν er endeligt μ -koncentreret gælder $\nu(\mathcal{R}) < \infty$

$$\text{Set } c = \nu(\mathcal{R}) / \delta$$

Lad $f \in \mathcal{V}$ være givet

$$\mu(|f| > c) \stackrel{\substack{\uparrow \\ \text{Markov Ugl.} \\ 3.9}}{\leq} \frac{1}{c} \int_{\mathcal{R}} |f| d\mu \leq \frac{1}{c} \nu(\mathcal{R}) = \delta$$

Hvis $f \in V \setminus S$:

$$\int_{\{f>c\}} |f| d\mu \stackrel{(*)}{\leq} \frac{\epsilon}{2} + \mu(\{f>c\}) \stackrel{(*)}{\leq} \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ idet } F := \{f>c\} \in \sigma(f), \mu(F) \leq \delta$$

Hvis $f \in S$:

S er en endelig delmængde af $L^1(\mu) \Rightarrow |f| < \infty$ μ -n.o. 3.3.6-7

$$\lim_{c \rightarrow \infty} \int_{\{f>c\}} |f| d\mu = \int_{\{f=\infty\}} |f| d\mu = 0, \text{ dvs. } \exists a_f \in \mathbb{R}_+ : \int_{\{f>a_f\}} |f| d\mu < \epsilon$$
3.3.1, 3.3.8

Set $a = \max\{c; a_f, f \in S\}$, da gælder

$$\forall f \in S: \int_{\{f>a\}} |f| d\mu \leq \int_{\{f>a_f\}} |f| d\mu < \epsilon$$
3.3.3

$\forall f \in V \setminus S$:

$$\int_{\{f>a\}} |f| d\mu \leq \int_{\{f>c\}} |f| d\mu < \epsilon$$
3.3.3

□

(4) $\Rightarrow$ (5)

At V er endeligt μ -koncentreret er antaget,
dette medfører, at $\nu(\Omega) < \infty$

Set $c = 2\nu(\Omega)$

Pr. antagelse $\exists \{a_n\}_{n \in \mathbb{N}_0}, a_n \uparrow \infty, a_0 = 0$:

$$\forall k \in \mathbb{N}_0, f \in V: \int_{\{f>a_k\}} |f| d\mu \leq 4^{-k} \frac{c}{2} \quad (*)$$

Definer $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\varphi(t) = 2^k t$, $t \in [a_k, a_{k+1}]$, $k \in \mathbb{N}_0$

φ er voksende og $\lim_{t \rightarrow \infty} \frac{\varphi(t)}{t} = \lim_{t \rightarrow \infty} 2^{k(t)} = \infty$ (desuden: φ er en borelfkt.)

Lad $f \in \mathcal{V}$ være vilkårlig

$$\int_{\mathbb{R}} \varphi(|f|) d\mu = \sum_{k=0}^{\infty} 2^k \int_{\{a_k \leq f < a_{k+1}\}} |f| d\mu \leq \sum_{k=0}^{\infty} 2^k \int_{\{a_k \leq |f|\}} |f| d\mu \leq \sum_{k=0}^{\infty} 2^k \cdot 4^{-k} \frac{c}{2} = \sum_{k=0}^{\infty} 2^{-k} \frac{c}{2} = c$$

(5) $\Rightarrow$ (1)

$\mathcal{V}$ er endeligt μ -koncentreret

$\Downarrow$

$\nu(\Omega) < \infty$ $\forall \varepsilon > 0 \exists F \in \mathcal{F}: \mu(F) < \infty$ $\wedge \nu(F^c) \leq \varepsilon/3$ (*)

Lad $\varepsilon > 0$, $\{F_n\}_{n \geq 1} \subseteq \mathcal{F}$, $F_n \downarrow \emptyset$ være givet

Pr. antagelse $\exists c \in \mathbb{R}_+$, $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ borelfkt.: $\lim_{t \rightarrow \infty} \frac{\varphi(t)}{t} = \infty$ $\wedge \int_{\mathbb{R}} \varphi(|f|) d\mu \leq c$, $f \in \mathcal{V}$

Dvs. vi kan vælge $t_0 \in \mathbb{R}_+$: $\varphi(t) \geq \frac{3\varepsilon}{2} t$, $t \geq t_0$

Lad $f \in \mathcal{V}$ være givet

$\forall n \geq 1$:

$$\int_{F_n} |f| d\mu \leq \int_{\substack{3.3.3 F^c \\ + 3.3.12}} |f| d\mu + \int_{F_n \cap F \cap \{|f| \leq t_0\}} |f| d\mu + \int_{\{|f| \geq t_0\}} |f| d\mu$$

$$\leq \underbrace{\nu(F^c)}_{3.3.3} + t_0 \cdot \mu(F_n \cap F) + \frac{\varepsilon}{3c} \cdot \int_{\mathbb{R}} \varphi(|f|) d\mu \leq \frac{\varepsilon}{3} + t_0 \cdot \mu(F_n \cap F) + \frac{\varepsilon}{3}$$

$\uparrow$
antagelse
+ *

$\Downarrow$

$$\nu(F_n) = \sup_{f \in \mathcal{V}} \int_{F_n} |f| d\mu \leq \frac{\varepsilon}{3} + t_0 \cdot \mu(F_n \cap F) + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} + t_0 \cdot \mu(F \cap F_n)$$

Da $\mu(F) < \infty$ følger af 1.3.2 og 1.4.3, at $\exists N \geq 1 \forall n \geq N: \mu(F \cap F_n) \leq \frac{\varepsilon}{t_0}$

Dvs. $\nu(F_n) < \varepsilon$, $n \geq N$.

□

Konvergenzrelationen

$$(\Omega, \mathcal{F}, \mu) \text{ @ } \mathbb{N}, \quad \varphi, \psi \in \mathcal{L}(0, \infty) \quad \exists C \in \mathbb{R}_+, \forall x \in \mathbb{R}_+: \psi(2x) \leq C \varphi(x)$$

$$f, \{f_n\}_{n \geq 1} \in M(\mu)$$

Da galder:

(1):

$$f_n \xrightarrow{n \rightarrow \infty} f \quad \mu\text{-u.o.}$$

$$a) \quad \{\varphi(|f_n - f|) \mid n \geq 1\} \text{ er unif. int.} \implies f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \varphi\text{-norm}$$

$$b) \quad \{\psi(|f_n|) \mid n \geq 1\} \text{ er unif. int.} \implies f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \psi\text{-norm}$$

(2):

$$\forall n \geq 1 \exists \varepsilon_n > 0 : \sum_{n=1}^{\infty} \mu(|f_n - f| > \varepsilon_n) < \infty \wedge \lim_{n \rightarrow \infty} \varepsilon_n = 0 \implies f_n \xrightarrow{n \rightarrow \infty} f \quad \mu\text{-u.o.}$$

(3):

$$f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \mu\text{-mal} \implies \exists \sigma(n) \in \mathbb{N}, \sigma(1) < \sigma(2) < \dots < \sigma(n) < \sigma(n+1) < \dots; f_{\sigma(n)} \xrightarrow{n \rightarrow \infty} f \quad \mu\text{-u.o.}$$

(4):

$$\sum_{n=1}^{\infty} \|f_n - f\|_{\varphi} < \infty \implies f_n \xrightarrow{n \rightarrow \infty} f \quad \mu\text{-u.o.}$$

(5):

$$f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \varphi\text{-norm} \implies f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \mu\text{-mal}$$

(6):

$$f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \mu\text{-mål}$$

$$a) \{ \varphi(|f_n - f|) | n \geq 1 \} \text{ er unif. int.} \Rightarrow f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \varphi\text{-norm}$$

$$b) \{ \psi(|f_n|) | n \geq 1 \} \text{ er unif. int.} \Rightarrow f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \psi\text{-norm}$$

(7):

$$f_n \xrightarrow{n \rightarrow \infty} f \text{ } \mu\text{-n.o.} \wedge \mu(\Omega) < \infty \Rightarrow f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \mu\text{-mål}$$

(8):

$$\exists M \in \mathbb{R}_+ : \varphi \leq M \wedge \mu(\Omega) < \infty :$$

$$f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \mu\text{-mål} \iff f_n \xrightarrow{n \rightarrow \infty} f \text{ i } \varphi\text{-norm}$$

(9):

$$\lim_{n, m \rightarrow \infty} \|f_n - f_m\|_\varphi = 0 \quad \exists h \in M(\mu) : f_n \xrightarrow{n \rightarrow \infty} h \text{ i } \varphi\text{-norm}$$

(10):

$$(L^q(\mu), \|\cdot\|_q) \text{ er et komplet pseudometrisk rum, } 0 \leq q \leq \infty$$

Beweis:

(1):

$$f_n \xrightarrow{n \rightarrow \infty} f \quad \mu\text{-n.o.}$$

a)

$\{\varphi(|f_n - f|) | n \geq 1\}$ er unif. int.

$$f_n \xrightarrow{n \rightarrow \infty} f \quad \mu\text{-n.o.} \xRightarrow[\substack{\uparrow \\ \text{egenskab ved } \varphi}]{\varphi} \varphi(|f_n - f|) \xrightarrow{n \rightarrow \infty} 0 \quad \mu\text{-n.o.}$$

Af 3.23.4 (Dunford-Pettis' Sætning) følger, da $\{\varphi(|f_n - f|) | n \geq 1\}$ er unif. int. og $\varphi(|f_n - f|) \xrightarrow{n \rightarrow \infty} 0 \quad \mu\text{-n.o.}$, at $f_n \xrightarrow{n \rightarrow \infty} f$ i φ -norm.

b)

$\{\psi(|f_n|) | n \geq 1\}$ er unif. int.

ψ er voksende og værstrekket, dvs. $\psi(|f|) \leq \liminf_{n \rightarrow \infty} \psi(|f_n|) \quad \mu\text{-n.o.} \quad (*)$
P. 4.23

$$\int \psi(|f|) d\mu \leq \int \liminf_{n \rightarrow \infty} \psi(|f_n|) d\mu \stackrel{3.3.3}{\leq} \liminf_{n \rightarrow \infty} \int \psi(|f_n|) d\mu \stackrel{3.5}{\leq} \liminf_{n \rightarrow \infty} \int \psi(|f_n|) d\mu \leq \sup_{n \geq 1} \int \psi(|f_n|) d\mu = V(\mathcal{A}) < \infty$$

Dvs. $\psi(|f|) \in L^1(\mu)$ af 3.22.37 følger, at $\{\psi(|f|)\}$ er unif. int.

$$\begin{aligned} \psi(|f_n - f|) &\stackrel{\psi \text{ vok.}}{\leq} \psi(|f_n| + |f|) \stackrel{\psi \text{ vok.}}{\leq} \psi(2 \cdot (|f_n| \vee |f|)) \stackrel{\text{antagelse om } \psi}{\leq} C \cdot \psi(|f_n| \vee |f|) \\ &\stackrel{\psi \text{ vok.}}{\leq} C \cdot (\psi(|f_n|) + \psi(|f|)) \quad , n \geq 1 \end{aligned}$$

Af 3.22.39 følger, idet $\{\psi(|f_n|) | n \geq 1\}, \{\psi(|f|)\}$ er unif. int. og $C \in \mathbb{R}_+$, at $\{\psi(|f_n - f|) | n \geq 1\}$ er unif. int. Da vi desuden har, at $f_n \xrightarrow{n \rightarrow \infty} f \quad \mu\text{-n.o.} \implies \psi(|f_n - f|) \xrightarrow{n \rightarrow \infty} 0 \quad \mu\text{-n.o.}$ følger af a), at $f_n \xrightarrow{n \rightarrow \infty} f$ i ψ -norm

□

(2):

$$\forall n \geq 1 \exists \varepsilon_n > 0 : \sum_{n=1}^{\infty} \mu(|f_n - f| > \varepsilon_n) < \infty \wedge \lim_{n \rightarrow \infty} \varepsilon_n = 0$$

Af 1.4.6 (Det 1. Borel-Cantelli lemma) følger,
at $\mu(\limsup_{n \rightarrow \infty} \{|f_n - f| > \varepsilon_n\}) = 0$

$$\text{Lad } \omega \in (\limsup_{n \rightarrow \infty} \{|f_n - f| > \varepsilon_n\})^c = \liminf_{n \rightarrow \infty} \{|f_n - f| \leq \varepsilon_n\}$$

Dvs. $\exists N \geq 1 \forall n \geq N : |f_n(\omega) - f(\omega)| \leq \varepsilon_n$, da $\varepsilon_n \xrightarrow{n \rightarrow \infty} 0$

har vi vist, at $f_n \xrightarrow{n \rightarrow \infty} f$, μ -u.o.

□

(3)

$f_n \xrightarrow{n \rightarrow \infty} f$ i μ -mål pr. def. 3.22.1 fås

$$\exists \sigma(n) \in \mathbb{N} : \sigma(1) < \sigma(2) < \dots < \sigma(n) < \sigma(n+1) < \dots$$

$$\forall n \geq \sigma(p), p \geq 1 : \mu(|f_n - f| > 2^{-p}) \leq 2^{-p}$$

$$\sum_{p=1}^{\infty} \mu(|f_{\sigma(p)} - f| > 2^{-p}) = \sum_{p=1}^{\infty} 2^{-p} = 1 < \infty, \quad 2^{-p} \xrightarrow{p \rightarrow \infty} 0 \text{ af (2) følger det ønskede}$$

nemlig, at $f_{\sigma(n)} \xrightarrow{n \rightarrow \infty} f$, μ -u.o.

□

(4)

$$\sum_{n=1}^{\infty} \|f_n - f\|_p = \sum_{n=1}^{\infty} \int \varphi(|f_n - f|) d\mu < \infty$$

$\Downarrow$ 3.7 Beppo-Levi

$$\sum_{n=1}^{\infty} \varphi(|f_n - f|) < \infty \quad \mu\text{-u.o.}$$

$\Downarrow$

$$\varphi(|f_n - f|) \xrightarrow{n \rightarrow \infty} 0 \quad \mu\text{-u.o.}$$

$\Downarrow$ φ voksende, $\varphi|_{x>0} > 0$

$$|f_n - f| \xrightarrow{n \rightarrow \infty} 0 \quad \mu\text{-u.o.}$$

□

(5):

Bewist i 3.22 (3.22.5)

(6):

$f_n \xrightarrow{n \rightarrow \infty} f$ i μ -mål

a)

$\{\varphi(|f_n - f|) | n \geq 1\}$ er uniformt integrabel

Bewis ved modstrid:

Antag, at $f_n \xrightarrow{n \rightarrow \infty} f$ i φ -norm er et falsket udsagn

$\exists \varepsilon_0 > 0, N \geq 1 \forall n \geq N: \|f_n - f\|_{\varphi} \geq \varepsilon_0$

$\exists \sigma(n) \geq N, n \geq 1, \sigma(1) < \sigma(2) < \dots : f_{\sigma(n)} \xrightarrow{n \rightarrow \infty} f$ μ -n.o. (jvf. (3))

$\{\varphi(|f_{\sigma(n)} - f|) | n \geq 1\} \subseteq \{\varphi(|f_n - f|) | n \geq 1\}$, og dermed er

$\{\varphi(|f_{\sigma(n)} - f|) | n \geq 1\}$ unif. int. af (1) følger, at $f_{\sigma(n)} \xrightarrow{n \rightarrow \infty} f$ i φ -norm, men dette er i modstrid med, at $\forall n \geq 1: \|f_{\sigma(n)} - f\|_{\varphi} \geq \varepsilon_0$

$\Downarrow$

Dvs. $f_n \xrightarrow{n \rightarrow \infty} f$ i φ -norm

□

b)

$\{\psi(|f_n|) | n \geq 1\}$ er uniformt integrabel

Bewis ved modstrid:

Antag, at $f_n \xrightarrow{n \rightarrow \infty} f$ i ψ -norm er et falsket udsagn

$\exists \varepsilon_0 > 0, N \geq 1 \forall n \geq N: \|f_n - f\|_{\psi} \geq \varepsilon_0$

$\exists \sigma(n) \geq N, n \geq 1, \sigma(1) < \sigma(2) < \dots : f_{\sigma(n)} \xrightarrow{n \rightarrow \infty} f$ μ -n.o. (jvf. (3))

$\{\psi(|f_{\sigma(n)}|) | n \geq 1\} \subseteq \{\psi(|f_n|) | n \geq 1\}$ og dermed er $\{\psi(|f_{\sigma(n)}|) | n \geq 1\}$ unif. int.

af (1b) følger, at $f_{\sigma(n)} \xrightarrow{n \rightarrow \infty} f$ i ψ -norm, men dette er i modstrid med, at $\forall n \geq 1: \|f_{\sigma(n)} - f\|_{\psi} \geq \varepsilon_0$

$\Downarrow$

Dvs. $f_n \xrightarrow{n \rightarrow \infty} f$ i ψ -norm

□

(7):

$$f_n \xrightarrow{n \rightarrow \infty} f \quad \mu\text{-n.o.} \wedge \mu(\Omega) < \infty$$

Lad $\varepsilon > 0$ være vilkårlig

$$\bigcup_{n=1}^{\infty} \{ |f_n - f| > \varepsilon \} \subseteq \Omega, \quad \mu(\Omega) < \infty \quad \text{af 1.4.5 følger}$$

$$0 \leq \limsup_{n \rightarrow \infty} \mu(|f_n - f| > \varepsilon) \leq \mu\left(\limsup_{n \rightarrow \infty} \{ |f_n - f| > \varepsilon \}\right) = \mu\left(\bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} \{ |f_n - f| > \varepsilon \}\right)$$

$$\begin{aligned} &= \mu\left(\omega \in \Omega : \{ |f_n(\omega) - f(\omega)| > \varepsilon \}_{n=1}^{\infty} \not\xrightarrow{n \rightarrow \infty} 0\right) \\ &\quad \uparrow \text{skriv evt. def. på konvergens op.} \quad \uparrow f_n \xrightarrow{n \rightarrow \infty} f \quad \mu\text{-n.o.} \end{aligned}$$

□

(8):

Bevist i 3.22 (3.22.9)

(9)+(10):

Vi antager, at $\{f_n\}_{n=1}^{\infty}$ er en Cauchyfølge mht. pseudometrikken $\|\cdot\|_0$

↓

$$\exists r(k) \geq 1, k \geq 1 \quad \forall k \geq 1: \sigma(k) < \sigma(r(k)) \quad \forall n, m \geq r(k): \|f_n - f_m\|_0 \leq 2^{-k}$$

$$\forall n, m \geq r(k), k \geq 1: \mu(|f_n - f_m| > 2^{-k}) \underset{3.22.18}{\leq} \|f_n - f_m\|_0 \leq 2^{-k} \quad (*)$$

$$\forall k \geq 1: F_k = \{ |f_{\sigma(k+1)} - f_{\sigma(k)}| > 2^{-k} \}, F_k^c = \bigcup_{j=k}^{\infty} F_j, N = \limsup_{k \rightarrow \infty} F_k$$

$$\sum_{k=1}^{\infty} \mu(F_k) = \sum_{k=1}^{\infty} \mu(|f_{\sigma(k+1)} - f_{\sigma(k)}| > 2^{-k}) \underset{(*)}{\leq} \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty \underset{\substack{1.4.6 \\ \text{Det. 1. BC-Lemma}}}{\implies} \mu(N) = 0$$

$$\text{Lad } \omega \in \Omega \setminus N = \liminf_{k \rightarrow \infty} F_k^c = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} F_k^c = \bigcap_{j=1}^{\infty} (F_j)^c \text{ være vilkårlig}$$

Dvs. $\exists m \geq 1, \omega \notin F^m$

$$\forall n > k \geq m: |f_{\sigma(n)}(\omega) - f_{\sigma(k)}(\omega)| \leq \sum_{j=k+1}^n |f_{\sigma(j)}(\omega) - f_{\sigma(j-1)}(\omega)| \leq \sum_{j=k+1}^n 2^{-j-1} \leq 2^{-k+1}$$

Dvs. $\{f_{\sigma(n)}(\omega)\}_{n=1}^{\infty}$ er en Cauchy-følge på $\mathbb{R} \implies \exists h(\omega) \in \mathbb{R}: \lim_{n \rightarrow \infty} f_{\sigma(n)}(\omega)$

$w \in \mathbb{R} \setminus \mathbb{N}$ var vilkårlig, def. vi $h: \mathbb{R} \rightarrow \mathbb{R}$ som ovenfor på $\mathbb{R} \setminus \mathbb{N}$,
 og sætter $h(w) = 0$ for $w \in \mathbb{N}$ har vi juf. 1.40.4 og def. 1.38.1
 (samt regneregler for σ -algebra og Urbilleder af fkt.), at $h \in M(\mu)$ og opfylder:

$$\lim_{j \rightarrow \infty} |f_n(w) - f_{\sigma(j)}(w)| = |f_n(w) - h(w)|, \quad w \in \mathbb{R} \setminus \mathbb{N}, \quad n \geq 1$$

$$\mu(|h - f_{\sigma(k)}| > 2^{-k+1}) \leq 2^{-k+1}, \quad k \geq 1 \quad \xRightarrow{3.22.11} \|h - f_n\|_0 \leq 2^{-k+1}, \quad k \geq 1$$

$$\Rightarrow \|h - f_n\|_0 \leq \|h - f_{\sigma(k)}\|_0 + \|f_{\sigma(k)} - f_n\| \leq 2^{-k+1} + 2^{-k}, \quad n \geq \sigma(k)$$

3.22.17

$$\Rightarrow f_n \xrightarrow{\mu} h \text{ i } \mu\text{-mål}$$

Da vi i 3.22.31 har vist, at $(L^0(\mu), \|\cdot\|_0)$ er et pseudo metrisk rum
 kan vi dermed konkludere, at $(L^0(\mu), \|\cdot\|_0)$ er et komplet pseudo-
 metrisk rum. (Dvs. vi har vist påstand (10) for $q=0$)

Antag $\lim_{n,m \rightarrow \infty} \|f_n - f_m\|_q = 0$, dvs. (juf. 3.22.4) at $\{f_n\}_{n \geq 1}$
 er en Cauchy-følge mht. $\|\cdot\|_0$ (og bn. m. s. 186), dvs.

$$\lim_{n,m \rightarrow \infty} \|f_n - f_m\|_0 = 0$$

Juf. det netop viste $f_{\sigma(k)} \geq 1, k \geq 1, h \in M(\mu)$ således, at

$$\lim_{k \rightarrow \infty} |f_n - f_{\sigma(k)}| = |f_n - h| \quad \mu\text{-n.o.}, \quad n \geq 1$$

$\Downarrow$ φ voksende, v. kont.

$$\varphi(|f_n - h|) \leq \liminf_{k \rightarrow \infty} \varphi(|f_n - f_{\sigma(k)}|) \quad \mu\text{-n.o.}, \quad n \geq 1$$

P.4.23

$\Downarrow$

$$\|f_n - h\|_q = \int_{\mathbb{R}} \varphi(|f_n - h|) d\mu \leq \int_{\mathbb{R}} \liminf_{k \rightarrow \infty} \varphi(|f_n - f_{\sigma(k)}|) d\mu$$

3.3.3

$$\leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}} \varphi(|f_n - f_{\sigma(k)}|) d\mu =: \liminf_{k \rightarrow \infty} \|f_n - f_{\sigma(k)}\|_q, \quad n \geq 1$$

2.5 Fatou's lemma

Af P.4.22 følger, at $0 \leq \lim_{n \rightarrow \infty} \|f_n - h\|_q \leq \lim_{n,k \rightarrow \infty} \|f_n - f_{\sigma(k)}\|_q = 0$

Dvs. $f_n \xrightarrow{\mu} h$ i φ -norm og dermed er (9) bevist

Da $\varphi: \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$, $t \in \overline{\mathbb{R}}_+ : \varphi(t) = t^q$, $q \in]0, \infty[$, $\varphi \in L(0, \infty)$
følger (10) af (9) for $q \in]0, \infty[$, da vi samtidig erindrer 3.22.31

$$\text{Antag } \lim_{n, m \rightarrow \infty} \|f_n - f_m\|_\infty = 0$$

$\Downarrow$

$$\exists N \in \mathbb{N}(\mu); k \geq 1 \forall w \in \Omega \cap \mathbb{N}; n, m \geq k : |f_n - f_m| < \varepsilon, \quad \varepsilon > 0$$

Dvs. $\{f_n(w)\}$, $w \in \Omega \cap \mathbb{N}$ er en Cauchy-følge på $\mathbb{R}$

$\Downarrow$

$$\exists h(w) \in \mathbb{R} : \lim_{n \rightarrow \infty} f_n(w) = h(w), \quad w \in \Omega \cap \mathbb{N}, \text{ s\u00e5t } h(w) = 0, \quad w \in \mathbb{N}$$

af 1.40.4 og 1.38.1 (samt regnearbejde for σ -alg. og ubilleder af fkt.) f\u00e5s
at h er en m\u00e5lelig fkt. og $\lim_{n \rightarrow \infty} f_n = h$ samt $\exists c \in \mathbb{R}_+ : h \leq c \mu$ -n.o.,
dvs. $h \in L^\infty(\mu)$ og $\lim_{n \rightarrow \infty} f_n = h$ alts\u00e5 er det pseudometriske rum (se 3.22.31)
 $(L^\infty(\mu), \|\cdot\|_\infty)$ komplet.

□

Lebesgueintegral på $\mathbb{R}$ og $\mathbb{R}^k$, k21 (3.26)

Lad λ betegne Lebesguemålt på $\mathbb{R}$

Lad $I = [a, b]$ betegne et begrænset interval (dvs. $a, b \in \mathbb{R}$), $f: I \rightarrow \mathbb{R}$

Påstand:

Hvis Riemannintegral $\int_a^b f(x) dx$ eksisterer, da eksisterer Lebesgueintegral $\int_I f d\lambda$ og $\int_a^b f(x) dx = \int_I f d\lambda$, dvs.

Lebesgueintegral er en udvidelse af Riemannintegral

Bevis:

Antag, at $\int_a^b f(x) dx$ eksisterer, dvs. (Lindstrøm 8.5.3 + s. 323-324)

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{j=1}^n m_{nj} = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{j=1}^n M_{nj} \quad \text{og } f \text{ er begrænset}$$

hvor $1 \leq j \leq n$: $m_{nj} := \inf \{f(t) \mid t \in I_{nj}\}$

$$M_{nj} = \sup \{f(t) \mid t \in I_{nj}\}$$

$$I_{nj} := \left[\frac{j-1}{n}(b-a) + a, \frac{j}{n}(b-a) + a \right], \quad J_{nj} := \left] \frac{j-1}{n}(b-a) + a, \frac{j}{n}(b-a) + a \right]$$

$$f_n := \sum_{j=1}^n m_{nj} 1_{I_{nj}}, \quad g_n := \sum_{j=1}^n M_{nj} 1_{J_{nj}}, \quad f_n \leq f \leq g_n \quad (*)$$

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{j=1}^n m_{nj} = \lim_{n \rightarrow \infty} \sum_{j=1}^n \lambda(I_{nj}) m_{nj} \stackrel{3.8.1}{=} \lim_{n \rightarrow \infty} \sum_{j=1}^n \int_{I_{nj}} m_{nj} d\lambda$$

$$= \lim_{n \rightarrow \infty} \int_I f_n d\lambda \stackrel{P.4.22}{=} \limsup_{n \rightarrow \infty} \int_I f_n d\lambda \leq \int_I \limsup_{n \rightarrow \infty} f_n d\lambda$$

$\uparrow$
uldt passende $h \in L^1(\lambda)$, $f_n \leq h$

$$\stackrel{3.3.3}{\leq} \int_I f d\lambda \stackrel{3.3.3}{\leq} \int_I \liminf_{n \rightarrow \infty} g_n d\lambda \stackrel{3.5}{\leq} \liminf_{n \rightarrow \infty} \int_I g_n d\lambda \stackrel{P.7.22}{=} \lim_{n \rightarrow \infty} \int_I g_n d\lambda$$

$$\stackrel{3.3.3}{=} \lim_{n \rightarrow \infty} \sum_{j=1}^n \int_{J_{nj}} M_{nj} d\lambda \stackrel{3.8.1}{=} \lim_{n \rightarrow \infty} \sum_{j=1}^n \lambda(J_{nj}) M_{nj} = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{j=1}^n M_{nj} = \int_a^b f(x) dx$$

□

$$(f * g) * h(v) = \int_{\mathbb{R}^k} (f * g)(x) h(v-x) \lambda_k(dx)$$

$$= \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} f(y) g(x-y) \lambda_k(dy) h(v-x) \lambda_k(dx)$$

$$\stackrel{\text{Bm. II 3.26.7}}{=} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} f(y) g(x-y) h(v-x) \lambda_k(dx) \lambda_k(dy)$$

$$= \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} f(y) g(x-y) h(v-y-(x-y)) \lambda_k(dx) \lambda_k(dy)$$

$$= \int_{\mathbb{R}^k} f(y) \int_{\{x-y | x \in \mathbb{R}^k\}} g(z) h(v-y-z) \lambda_k(dz) \lambda_k(dy)$$

†

$$= \int_{\mathbb{R}^k} f(y) (g * h)(v-y) \lambda_k(dy) = f * (g * h)(v)$$

Den diskrete faldning:

Lad $v \in \mathbb{R}^k$ være vilkårlig

$$f * g(v) = \sum_{x \in \mathbb{R}^k} f(x) g(v-x) = \sum_{\{z=v-x | x \in \mathbb{R}^k\}} f(v-z) g(z) = \sum_{z \in \mathbb{R}^k} g(z) f(v-z) = g * f(v)$$

$$(f * g) * h(v) = \sum_{x \in \mathbb{R}^k} \left(\sum_{y \in \mathbb{R}^k} f(y) g(x-y) \right) h(v-x)$$

$$= \sum_{x \in \mathbb{R}^k} \left(\sum_{y \in \mathbb{R}^k} f(y) g(x-y) h(v-y-(x-y)) \right)$$

$$\stackrel{\text{sum pos. l'ed}}{=} \sum_{y \in \mathbb{R}^k} f(y) \left(\sum_{x \in \mathbb{R}^k} g(x-y) h(v-y-(x-y)) \right)$$

$$= \sum_{y \in \mathbb{R}^k} f(y) \left(\sum_{\{z=x-y | x \in \mathbb{R}^k\}} g(z) h(v-y-z) \right) = \sum_{y \in \mathbb{R}^k} f(y) \left(\sum_{z \in \mathbb{R}^k} g(z) h(v-y-z) \right)$$

$$= \sum_{y \in \mathbb{R}^k} f(y) (g * h)(v-y) = f * (g * h)(v)$$

□

Notation:

$$\forall a, b \in \mathbb{R}: \int_a^b f(x) dx := \int_{[a, b]} f d\lambda, \quad 1_{[a, b]} f \in L(\lambda)$$

$$\forall A \in \mathcal{B}(\mathbb{R}): \int_A f(x) dx := \int_A f d\lambda, \quad 1_A f \in L(\lambda)$$

For $f: \mathbb{R}^k \rightarrow \mathbb{R}$ gælder som for reelle funktioner af en reel variable, at hvis f er Riemannintegrabel, så er f Lebesgueintegrabel. Vi indfører følgende notation for:

$$\int_A f(x) dx := \int_A f d\lambda_k, \quad \forall A \in \mathcal{B}(\mathbb{R}^k), 1_A f \in L(\lambda_k)$$

$$\int_A f(x_1, x_2, x_3, \dots, x_k) dx_1 dx_2 dx_3 \dots dx_k := \int_A f d\lambda_k, \quad \forall A \in \mathcal{B}(\mathbb{R}^k), 1_A f \in L(\lambda_k)$$

$$\text{Påstand: } \lambda_k = \underbrace{\lambda \otimes \lambda \otimes \dots \otimes \lambda}_{k \text{ gange}} \quad (*)$$

Bevis: 1.33.5 + induktion på 3.20.8 $\square$

$$\text{Påstand: } 1_A f \in L(\lambda_k)$$

Integral af $1_A f$ kan findes ved, at integrerer mlt. til en koordinat ad gangen (rækkefølge ligegyldig).

Bevis: $(*) + 3.21$ (Tonelli-Fubini's Sætning) $\square$

Eksempel på at Lebesgueintegral er en udvidelse af Riemannintegral

$$I = [0, 1], \quad f: I \rightarrow \{0, 1\}$$

$$\forall x \in I: f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

f er ikke integrabel, idet hver øvre sum er 1 og hver nedre sum er 0.

f er Lebesgueintegrabel, idet $f = 1_{\mathbb{Q} \cap [0, 1]}$

$$\int_0^1 f(x) dx := \int_{[0, 1]} 1_{\mathbb{Q}} d\lambda = \lambda(\mathbb{Q} \cap [0, 1]) \stackrel{\substack{\uparrow \\ \mathbb{Q} \text{ tællig}}}{=} \sum_{q \in \mathbb{Q} \cap [0, 1]} \lambda(\{q\}) \stackrel{\substack{\uparrow \\ \text{Ex. 2.34 (ii)} \\ F(x) = x}}{=} 0$$

Påstand: $f: [a, b] \rightarrow \mathbb{R}$ er Riemannintegrabel hvis
 f er begrænset og kontinuert λ -n.o.

Er ikke bevist i Hoffmann-Jørgensen

Lebesgue-Stieltjeintegralet (3.27)

$F: \mathbb{R} \rightarrow \mathbb{R}$ er voksende og højrekontinuerligt

Notation:

$$\forall A \in \mathcal{B}(\mathbb{R}), 1_A f \in L(\lambda_F): \int_A f(x) F(dx) = \int_A f dF := \int_A f d\lambda_F$$
$$\int_a^b f(x) F(dx) = \int_a^b f dF = \int_A f dF, A =]a, b] \cap \mathbb{R}$$

Påstand:

$$\varphi: \mathbb{R} \rightarrow \overline{\mathbb{R}}_+, \varphi \text{ måltlig og } \forall a \leq b: F(b) - F(a) = \int_a^b \varphi(x) dx$$

$$\text{Da gælder } \int_A f(x) F(dx) = \int_A f(x) \varphi(x) dx, A \in \mathcal{B}(\mathbb{R})$$

(Hvis et af integralerne eksisterer, eksisterer det andet, og de er ens)

Bevis:

Jvf. 3.17 er $\nu: \mathcal{B}(\mathbb{R}) \rightarrow \overline{\mathbb{R}}_+, \forall A \in \mathcal{B}(\mathbb{R}): \nu(A) = \int_A \varphi(x) dx$ et mål på $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Jvf. 1.32.2, 1.10.1 og 1.7 fås, at $\lambda_F = \nu$ på $\mathcal{B}(\mathbb{R})$.

Da $\lambda_F(A) = \int_A \varphi(x) dx = \int_A \varphi d\lambda, \forall A \in \mathcal{B}(\mathbb{R})$ følger (jvf. def. i 3.17), at

λ_F er absolut λ -kontinuerligt (i kort notation $\lambda_F \ll \lambda$)

og λ -tætheden af λ_F er φ ($\frac{d\lambda_F}{d\lambda} = \varphi$)

Det ønske står nu direkte at læse af 3.17.7

□

$F: I \rightarrow \mathbb{R}$, I interval på reelle akse, F voksende højrekont.
 $\varphi: I \rightarrow \mathbb{R}$ Borelfkt.

F kaldes absolut kontinuert og φ kaldes tætheden for F , hvis
$$F(b) - F(a) = \int_a^b \varphi(x) dx, \forall a \leq b, a, b \in I$$

Påstand (framsættes uden bevis dog med henvisning til Exc. 3.37):

Hvis F er absolut kontinuert med tæthed φ , da gælder:

$\exists N \in \mathcal{N}(\lambda), N \subseteq I: F'(x) = \varphi(x), x \in I \setminus N$ (dvs. F diff. på $I \setminus N$ med afl. φ)

Hvis F er diff. på $I \setminus T$, T er endtelig delmængde af I

og hvis $(1_{I \setminus T} F')^+ \in L(\lambda)$ eller $(1_{I \setminus T} F')^- \in L(\lambda)$, da gælder:

F er absolut kontinuert med tæthed $\varphi(x) = F'(x), x \in I \setminus T$

Påstand (framsættes uden bevis dog med henvisning til Exc. 3.3):

Antag at F er absolut kontinuert med tæthed φ .

Hvis φ er kontinuert i a , da er F diff. i a og $F'(a) = \varphi(a)$

Hvis φ er diff. i a , da er F 2-gangs diff. i a og $F''(a) = \varphi'(a)$

Lad $F: \mathbb{R}^k \rightarrow \mathbb{R}$ være højrekontinueret med positive tilvækster

Notation:

$$\int_A f dF = \int_A f(x) F(dx) = \int_A f(x_1, \dots, x_k) F(dx_1, \dots, dx_k) := \int_A f d\lambda_F, \forall A \in \mathcal{B}(\mathbb{R}^k), 1_A f \in L(\lambda_F)$$

Ved et bevis svarende til beviset på forrige side (med henvisning til 1.33.4 og 1.10.2 istedet for 1.32.2 og 1.10.1) finder vi:

Påstand:

$$\varphi: \mathbb{R}^k \rightarrow \mathbb{R}_+, \varphi \text{ måltlig } \forall a, b \in \mathbb{R}^k, a \leq b: \Delta_a^b F = \int_{a \leq b} \varphi d\lambda_k$$

$$\text{Da gælder } \int_A f(x) F(dx) = \int_A f(x) \varphi(x) dx, A \in \mathcal{B}(\mathbb{R})$$

(Hvis et af integralene eksisterer, eksisterer det andet, og de er ens)

og λ_F er absolut λ_k -kontinueret og φ er λ_k -tætheden for λ_F

$F: \mathbb{R}^k \rightarrow \mathbb{R}$, $\varphi: \mathbb{R}^k \rightarrow \mathbb{R}$, φ Borelfk.

F kaldes absolut kontinuert og φ kaldes tætheden for F , hvis

$$\Delta_a^b F = \int_{a \leq b} \varphi d\lambda_k, \forall a, b \in \mathbb{R}^k, a \leq b$$

Påstand (fremsættes uden bevis):

Hvis $F: \mathbb{R}^k \rightarrow \mathbb{R}$ er kontinuert med positive tilvækster og hvis

$$\varphi(x) = \frac{\partial^k F}{\partial x_1 \dots \partial x_k}(x) \text{ eksisterer } \forall x \in \mathbb{R}^k \setminus N, N \subseteq \mathbb{R}^k \text{ tællig,}$$

da er F absolut kontinuert med tæthed φ

$F: \mathbb{R}^k \rightarrow \mathbb{R}$ højrebev. pos. tilværdster, $g: \mathbb{R}^k \rightarrow \overline{\mathbb{R}}_+$ Borelfkt.

$\forall v \in \mathbb{R}^k$: Lebesgue-Stieltjefoldningen af g og F : $g * F(v) = \int_{\mathbb{R}^k} g(v-x) F(dx)$

$f, g: \mathbb{R}^k \rightarrow \overline{\mathbb{R}}_+$ Borelfkt.

$\forall v \in \mathbb{R}^k$: Lebesguefoldningen af g og f : $g * f(v) = \int_{\mathbb{R}^k} f(x) g(v-x) dx$

Den diskrete foldning af g og f : $g * f(v) = \sum_{x \in \mathbb{R}^k} f(x) g(v-x)$

Påstand:

For Lebesguefoldningen og den diskrete foldning gælder:

$f * g = g * f$, $(f * g) * h = f * (g * h)$; $f, g, h: \mathbb{R}^k \rightarrow \overline{\mathbb{R}}_+$ Borelfkt.

(Et tilsvarende resultat gælder for Lebesgue-Stieltjefoldningen hvis f, g, h v. ford.fkt.)

Beweis:

Lebesguefoldningen:

Lad $v \in \mathbb{R}^k$ være vilkårlig

$T: \mathbb{R}^k \rightarrow \mathbb{R}^k$, $\forall x \in \mathbb{R}^k: T(x) = v - x$, T er klart en Borelfkt.

Ifølge sætn. 1.7 og 1.10.2 ser vi $\lambda_k = \lambda_{k_T}$

$$\begin{aligned} f * g(v) &= \int_{\mathbb{R}^k} f(x) g(v-x) \lambda_k(dx) = \int_{\mathbb{R}^k} f(v-x) g(x) \lambda_{k_T}(dx) \\ &= \int_{\mathbb{R}^k} f(v-x) g(x) \lambda_k(dx) = \int_{\mathbb{R}^k} g(x) f(v-x) \lambda_k(dx) = g * f(v) \end{aligned}$$

Jvf. Den første transformationsætningen (3.15)

Partiel integration (3.28)

$F, G: \mathbb{R} \rightarrow \mathbb{R}$; F, G er voksende og højrekont.

$$G(x-) := \lim_{y \uparrow x} G(y) = \sup_{y < x} G(y) \quad ; \quad H(x) := F(x)G(x), \quad x \in \mathbb{R}$$

Påstand:

$$-\infty \leq a \leq b \leq \infty:$$

$$\int_a^b (F(x) - F(a))G(dx) = \int_a^b (G(b) - G(y-))F(dy)$$

$$-\infty < a \leq b < \infty:$$

$$\int_a^b F(x)G(dx) + \int_a^b G(y)F(dy) = H(b) - H(a) \quad (*)$$

$$-\infty = a \leq b < \infty \vee -\infty = a \leq b = \infty \vee -\infty < a \leq b = \infty:$$

Hvis $\int_a^b F(x)G(dx) + \int_a^b G(y)F(dy)$ eksisterer (ikke er på formen $\infty - \infty$) eksisterer grænseværdierne $\lim_{x \rightarrow a} H(x) = H(a)$, $\lim_{x \rightarrow b} H(x) = H(b)$ og

$$\int_a^b F(x)G(dx) + \int_a^b G(y)F(dy) = H(b) - H(a)$$

Bemærkning:

Antag at F, G er absolut kontinuert med tætheder f, g ; $-\infty \leq a \leq b \leq \infty$

Juf. 3.27.5: $f = F'$, $g = G'$ λ -n.o. og 3.27.3 giver $G(y-) = G(y)$, $y \in \mathbb{R}$

$$3.27.4 + 3.3.4: \int_a^b F(x)G(dx) = \int_a^b F(x)G'(x)dx, \quad \int_a^b G(y)F(dy) = \int_a^b G(y)F'(y)dy$$

Hvis $\int_a^b F(x)G(dx) + \int_a^b G(y)F(dy)$ eksisterer (ikke er på formen $\infty - \infty$) for vi dermed:

$$\int_a^b F(x)G'(x)dx + \int_a^b G(x)F'(x)dx = H(b) - H(a) = [F(x)G(x)]_a^b$$

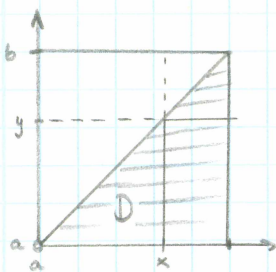
Beweis:

$$-\infty \leq a \leq b \leq \infty$$

Valg $\{a_n\}_{n \geq 1}, \{b_n\}_{n \geq 1} \in \mathbb{R} :]a_n, b_n] \uparrow]a, b]$

$$\left. \begin{aligned} \text{Af 1.32.2 og 1.4.2 følger } \lambda_F([a, b]) &= \lim_{n \rightarrow \infty} \lambda_F([a_n, b_n]) = \lim_{n \rightarrow \infty} (F(b_n) - F(a_n)) \\ &= F(b) - F(a) \text{ og tilsvarende, at } \lambda_G([a, b]) = G(b) - G(a) \\ \text{og (v.h.j.a. Exc 2.34 og valg af passende følger samt 1.4.2)} \\ \lambda_F([a, x]) &= F(x) - F(a), x \leq b \quad \lambda_G([y, b]) = G(b) - G(y-), y \leq b \end{aligned} \right\} (+)$$

$$D := \{(x, y) \in \mathbb{R}^2 \mid a < y \leq x \leq b\}$$



$$\int_a^b (F(x) - F(a)) G(dx) \stackrel{3.8.1 (+)}{=} \int_a^b G(dx) \int_a^b 1_D(x, y) F(dy) \stackrel{3.21.1 \text{ Tonellis s\ae tning}}{=} \int_a^b F(dy) \int_a^b 1_D(x, y) G(dx) \stackrel{3.8.1 (+)}{=} \int_a^b (G(b) - G(y-)) F(dy)$$

$$-\infty < a \leq b < \infty:$$

$$\begin{aligned} \int_a^b (F(x) - F(a)) G(dx) &\stackrel{3.3.12}{=} \int_a^b F(x) G(dx) - \int_a^b F(a) G(dx) = \int_a^b F(x) G(dx) - F(a) (G(b) - G(a)) \\ &= \int_a^b (G(b) - G(y-)) F(dy) \stackrel{3.3.12}{=} \int_a^b G(b) F(dy) - \int_a^b G(y-) F(dy) = G(b) (F(b) - F(a)) - \int_a^b G(y-) F(dy) \end{aligned}$$

$$\Downarrow \int_a^b F(x) G(dx) + \int_a^b G(y-) F(dy) = F(a) G(b) - F(a) G(a) + G(b) F(b) - G(b) F(a) = H(b) - H(a)$$

$$-\infty = a \leq b < \infty \vee -\infty < a \leq b < \infty:$$

$$\text{Vælg } \{a_n\}_{n \geq 1}, \{b_n\}_{n \geq 1} :]a_n, b_n] \uparrow]a, b] \Rightarrow]a_n, b] \uparrow]a, b],]a, b_n] \uparrow]a, b]$$

$$\text{Pr. antagelse eksisterer } c \in \overline{\mathbb{R}} : \lim_{n \rightarrow \infty} \left(\int_{a_n}^{b_n} F(x)G(dx) + \int_{a_n}^{b_n} G(y-)F(dy) \right) = c \Rightarrow \exists c_a, c_b \in \overline{\mathbb{R}}:$$

$$\lim_{n \rightarrow \infty} \left(\int_{a_n}^b F(x)G(dx) + \int_{a_n}^b G(y-)F(dy) \right) = c_b, \quad \lim_{n \rightarrow \infty} \left(\int_a^{b_n} F(x)G(dx) + \int_a^{b_n} G(y-)F(dy) \right) = c_a$$

$$\text{Juf. (*) fås: } \lim_{n \rightarrow \infty} (H(b) - H(a_n)) = H(b) - \lim_{n \rightarrow \infty} H(a_n) = c_b$$

$$\lim_{n \rightarrow \infty} (H(b_n) - H(a)) = \lim_{n \rightarrow \infty} H(b_n) - H(a) = c_a$$

↓

$$H(a) = \lim_{n \rightarrow \infty} H(a_n) = H(b) - c_b, \quad H(b) = \lim_{n \rightarrow \infty} H(b_n) = H(a) + c_a$$

Hvis $-\infty = a \leq b < \infty \vee -\infty < a \leq b = \infty$ følger dermed, at

hh. $H(-\infty)$ el. $H(\infty)$ eksisterer

Hvis $-\infty = a \leq b = \infty$ følger juf. (*) og ovenstående.

$$\forall n \geq 1: \underbrace{\int_a^{b_n} F(x)G(dx) + \int_a^{b_n} G(y-)F(dy)}_{\downarrow c_a} + \underbrace{H(-\infty)}_{\downarrow H(-\infty)} = H(b_n)$$

$$\text{Dvs. } H(\infty) = \lim_{n \rightarrow \infty} H(b_n) = c_a + H(-\infty) \in \overline{\mathbb{R}}$$

□

Substitution (3.29)

$F: \mathbb{R} \rightarrow \mathbb{R}$, F er voksende og højrekont. ; $\alpha = F(-\infty)$ $\beta = F(\infty)$

$\hat{F}:]\alpha, \beta[\rightarrow \mathbb{R}$; $\hat{F}(y) := \inf \{v \in \mathbb{R} \mid F(v) \geq y\}$ $y \in]\alpha, \beta[$

Juf. def. på α, β og da F er voksende og højrekont. følger, at $\hat{F}$ er veldef.

Påstand:

$\hat{F}$ er en voksende, venstre kontinuert funktion

$\forall x \in \mathbb{R}, y \in]\alpha, \beta[: F(x) \geq y \iff x \geq \hat{F}(y)$, $F(\hat{F}(y)-) \leq y \leq F(\hat{F}(y))$

$$\int_{-\infty}^{\infty} h(x) F(dx) = \int_{\alpha}^{\beta} h(\hat{F}(u)) du, \quad h: \mathbb{R} \rightarrow \mathbb{R} \text{ Borelfkt.}$$

$$\int_{-\infty}^{\infty} g(F(x)) F(dx) = \int_{\alpha}^{\beta} g(v) dv, \quad \text{hvis } F \text{ er kontinuert, } g:]\alpha, \beta[\rightarrow \mathbb{R} \text{ Borelfkt.}$$

Hvis F er absolut kontinuert med tæthedsfunktion f :

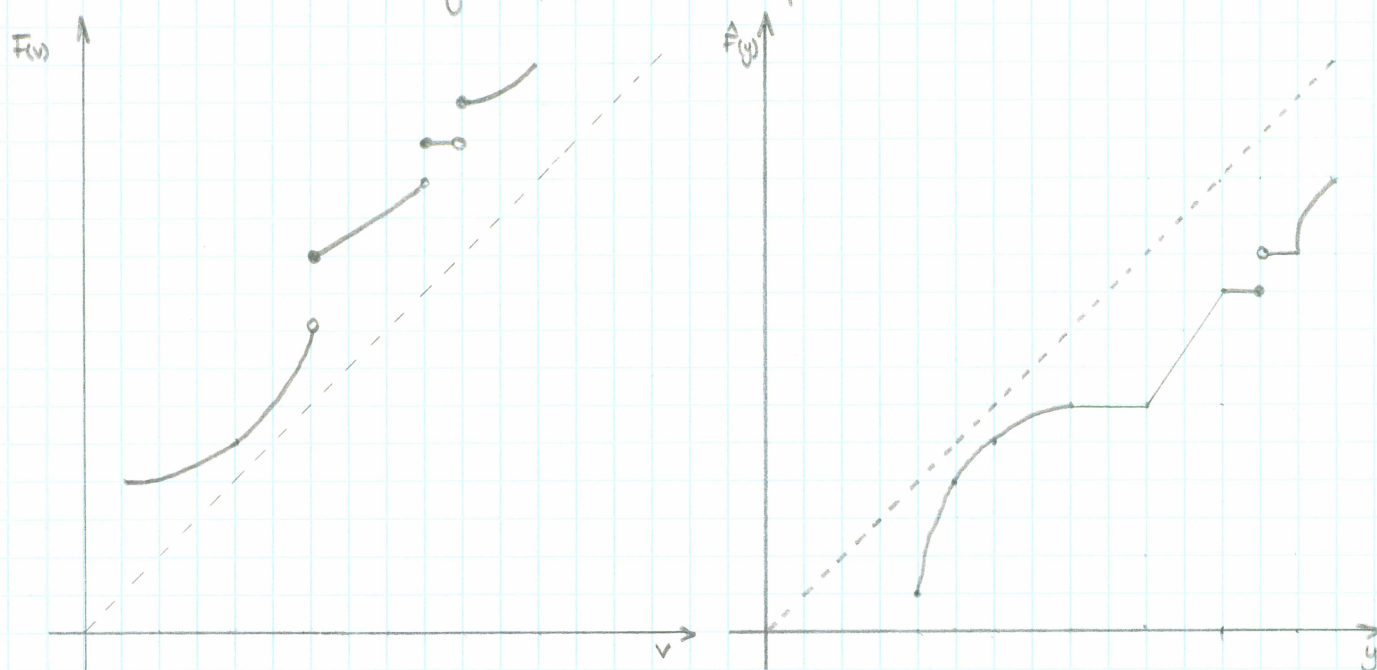
$$f = F' \text{ Lebesgue-n.o. og } \int_{-\infty}^{\infty} g(F(x)) F'(x) dx = \int_{\alpha}^{\beta} g(u) du$$

Hvis $g:]\alpha, \beta[\rightarrow \mathbb{R}$ er voksende og $h:]\alpha, \beta[\rightarrow \mathbb{R}$ er aftagende:

$$\int_{-\infty}^{\infty} g(F(x-)) F(dx) \leq \int_{\alpha}^{\beta} g(u) du \leq \int_{-\infty}^{\infty} g(F(x)) F(dx)$$

$$\int_{-\infty}^{\infty} h(F(x)) F(dx) \leq \int_{\alpha}^{\beta} h(u) du \leq \int_{-\infty}^{\infty} h(F(x-)) F(dx)$$

Da F er voksende, følger af definitionen på $\hat{F}$, at $\hat{F}$ er voksende.



Da F er højrekontinueret følger, at $\hat{F}$ er venstrekontinueret; vi har:

$$\forall x \in \mathbb{R}, y \in]\alpha, \beta[: F(x) \geq y \iff x \geq \hat{F}(y), \quad F(\hat{F}(y)-) \leq y \leq F(\hat{F}(y)) \quad (*)$$

Lad λ betegne Lebesguemålet på $]\alpha, \beta[$, $\forall B \in \mathcal{B}(]\alpha, \beta[)$ $\nu(B) := \lambda(\hat{F}^{-1}(B))$

Lad $a, b \in]\alpha, \beta[$, sæt $B =]a, b]$

$$\hat{F}^{-1}(B) = \{y \in]\alpha, \beta[\mid a < \hat{F}(y) \leq b\} = \{y \in]\alpha, \beta[\mid F(a) < y \leq F(b)\}$$

$$\nu(]a, b]) = \nu(\hat{F}^{-1}(B)) = \lambda(\hat{F}^{-1}(B)) = F(b) - F(a), \quad \text{dvs. } \nu = \lambda_F$$

$$\text{Af 3.15.1 følger } \int_{-\infty}^{\infty} h(x) F(dx) = \int_{\alpha}^{\beta} h(\hat{F}(u)) du$$

Antag, at F er kontinuert; af (*) følger, at for $h: \mathbb{R} \rightarrow \mathbb{R}$, $\forall x \in \mathbb{R}: h(x) = g(F(x))$

$$\text{gælder } \forall u \in]\alpha, \beta[: g(u) = h(\hat{F}(u))$$

$$\text{Af det ovenstående følger } \int_{-\infty}^{\infty} g(F(x)) F(dx) = \int_{\alpha}^{\beta} g(u) du$$

Hvis F er absolut kontinuert med tæthedsfkt. f :

$f = F'$ Lebesgue-n.o. juf. 3.27.5

$$\int_{-\infty}^{\infty} g(F(x)) F(dx) \underset{3.27.4}{=} \int_{-\infty}^{\infty} g(F(x)) f(x) dx \underset{3.3.4}{=} \int_{-\infty}^{\infty} g(F(x)) F'(x) dx = \int_{-\infty}^{\infty} g(u) du$$

↑
det foregående
3.29.4

Antag $g, h: [\alpha, \beta] \rightarrow \mathbb{R}$ er hh. voksende og aftagende

Af 3.29.2 (se ovenstående bevis) følger:

$$g(F(\hat{F}(y)-)) \leq g(y) \leq g(F(\hat{F}(y))) ; \quad h(F(\hat{F}(y)-)) \geq h(y) \geq h(F(\hat{F}(y))), y \in]\alpha, \beta$$

|| 3.3.3

$$\int_{\alpha}^{\beta} g(F(\hat{F}(y)-)) dy \leq \int_{\alpha}^{\beta} g(y) dy \leq \int_{\alpha}^{\beta} g(F(\hat{F}(y))) dy ; \quad \int_{\alpha}^{\beta} h(F(\hat{F}(y)-)) dy \geq \int_{\alpha}^{\beta} h(y) dy \geq \int_{\alpha}^{\beta} h(F(\hat{F}(y))) dy$$

|| 3.29.3

$$\int_{-\infty}^{\infty} g(F(x-)) dx \leq \int_{\alpha}^{\beta} g(y) dy \leq \int_{-\infty}^{\infty} g(F(x)) dx ; \quad \int_{-\infty}^{\infty} h(F(x-)) dx \geq \int_{\alpha}^{\beta} h(y) dy \geq \int_{-\infty}^{\infty} h(F(x)) dx$$

□

Abstrakte integraler som Lebesgue- / Lebesgue-Stieltjesintegraler (3.30)

$(\Omega, \mathcal{F}, \mu) \text{ @m}$, $f: \Omega \rightarrow \bar{\mathbb{R}}$, f måltlig ; μ endelig (eller nul på pkt.-mgd.)

$F: \mathbb{R} \rightarrow \mathbb{R}$ er en voksende højrekontinueret funktion.

$R, \tilde{R}: \mathbb{R} \rightarrow \bar{\mathbb{R}}$

$$R(x) = \begin{cases} \mu(f > x), & x \geq 0 \\ -\mu(f \leq x), & x < 0 \end{cases} ; \quad \tilde{R}(x) = \begin{cases} \mu(f \geq x), & x \geq 0 \\ -\mu(f < x), & x < 0 \end{cases}$$

Påstand:

$$\int_{\Omega} f d\mu = \int_{-\infty}^{\infty} R(t) dt = \int_{-\infty}^{\infty} \tilde{R}(t) dt, \quad \int_{\Omega} (F(f(\omega)) - F(0)) \mu(d\omega) = \int_{-\infty}^{\infty} \tilde{R}(x) F(dx)$$

Hvis μ er sum-endelig, $f \geq 0$: $\mu \otimes \lambda(G) = \int_{\Omega} f d\mu$, $G = \{(\omega, x) \mid 0 \leq x \leq f(\omega)\}$

Beweis:

$\forall x \in \mathbb{R}_+$: $\text{int}^+(x) :=$ mindste hele tal strengt større end x

$\forall x \in \mathbb{R}_-$: $\text{int}^-(x) :=$ største hele tal strengt mindre end x

$\forall n \geq 1, t \in \mathbb{R}_+$: $\psi_n^+(t) = 2^{-n} \cdot \text{int}^+(2^n \cdot t)$, $\forall n \geq 1, t \in \mathbb{R}_-$: $\psi_n^-(t) = 2^{-n} \cdot \text{int}^-(2^n \cdot t)$

$\forall n \geq 1, t \in \mathbb{R}_+$: $\psi_{n+1}^+(t) = 2^{-1} \cdot 2^{-n} \cdot \text{int}^+(2 \cdot 2^n \cdot t) \leq 2^{-n} \cdot \text{int}^+(2^n \cdot t) = \psi_n^+(t)$

$$t = 2^{-n} \cdot 2^n \cdot t < 2^{-n} \cdot \text{int}^+(2^n \cdot t) = \psi_n^+(t) \leq 2^{-n} (2^n \cdot t + 1) = t + 2^{-n}$$

$$\Downarrow \forall t \in \mathbb{R}_+: \psi_n^+(t) \downarrow t, n \rightarrow \infty$$

$\forall n \geq 1, t \in \mathbb{R}_-$: $\psi_{n+1}^-(t) = 2^{-1} \cdot 2^{-n} \cdot \text{int}^-(2 \cdot 2^n \cdot t) \geq 2^{-n} \cdot \text{int}^-(2^n \cdot t) = \psi_n^-(t)$

$$t = 2^{-n} \cdot 2^n \cdot t > 2^{-n} \cdot \text{int}^-(2^n \cdot t) \geq 2^{-n} (2^n \cdot t - 1) = t - 2^{-n}$$

$$\Downarrow \forall t \in \mathbb{R}_-: \psi_n^-(t) \uparrow t, n \rightarrow \infty$$

$\forall n \geq 1, t \in \mathbb{R}_+$: $R_n^+(t) := \sum_{j=1}^{\infty} R(j \cdot 2^{-n}) \cdot 1_{[j \cdot 2^{-n}, (j+1) \cdot 2^{-n})}(t)$; Bm.: $R_n^+(t) = R(\psi_n^+(t))$

$\forall n \geq 1, t \in \mathbb{R}_-$: $R_n^-(t) := \sum_{j=1}^{\infty} R(-j \cdot 2^{-n}) \cdot 1_{[-j \cdot 2^{-n}, -(j-1) \cdot 2^{-n})}(t)$; Bm.: $R_n^-(t) = R(\psi_n^-(t))$

Af ovenstående fås:

$$\forall t \in \mathbb{R}_+ : \{f > \psi_n^+(t)\} \uparrow \{f > t\}, n \rightarrow \infty \xRightarrow{1.4.2} R_n^+(t) = R(\psi_n^+(t)) \uparrow R(t), n \rightarrow \infty$$

$$\forall t \in \mathbb{R}_- : \{f \leq \psi_n^-(t)\} \uparrow \{f \leq t\}, n \rightarrow \infty \Rightarrow R_n^-(t) = R(\psi_n^-(t)) \uparrow R(t), n \rightarrow \infty$$

$$\begin{aligned} \int_0^\infty R(t) dt &\stackrel{3.4}{=} \lim_{n \rightarrow \infty} \int_0^\infty R_n^+(t) dt \stackrel{3.8.2}{=} \lim_{n \rightarrow \infty} \left\{ \sum_{j=1}^\infty R(j \cdot 2^{-n}) \cdot \lambda([j \cdot 2^{-n}, (j+1) \cdot 2^{-n}]) \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ 2^{-n} \cdot \sum_{j=1}^\infty \mu(f > j \cdot 2^{-n}) \right\} = \lim_{n \rightarrow \infty} \left\{ 2^{-n} \cdot \sum_{j=1}^\infty \mu(f^+ > j \cdot 2^{-n}) \right\} =: \int_{\mathbb{R}} f^+ d\mu \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^0 R(t) dt &\stackrel{3.4}{=} \lim_{n \rightarrow \infty} \int_{-\infty}^0 R_n^-(t) dt \stackrel{3.8.2}{=} \lim_{n \rightarrow \infty} \left\{ \sum_{j=1}^\infty R(-j \cdot 2^{-n}) \cdot \lambda([-j \cdot 2^{-n}, -(j-1) \cdot 2^{-n}]) \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ 2^{-n} \cdot \sum_{j=1}^\infty -\mu(f \leq -j \cdot 2^{-n}) \right\} = -\lim_{n \rightarrow \infty} \left\{ 2^{-n} \cdot \sum_{j=1}^\infty \mu(f^- \geq j \cdot 2^{-n}) \right\} \\ &= -\lim_{n \rightarrow \infty} \left\{ 2^{-n} \cdot \sum_{j=1}^\infty \mu(f^- > j \cdot 2^{-n}) \right\} - \lim_{n \rightarrow \infty} \left\{ 2^{-n} \cdot \sum_{j=1}^\infty \mu(f^- = j \cdot 2^{-n}) \right\} \\ &\quad \text{def. pos. Maß} \\ &\quad \text{+ nicht auf pos. reeller} \\ &= - \int_{\mathbb{R}} f^- d\mu \end{aligned}$$

$$\int_{-\infty}^\infty R(t) dt = \int_0^\infty R(t) dt + \int_{-\infty}^0 R(t) dt = \int_{\mathbb{R}} f^+ d\mu - \int_{\mathbb{R}} f^- d\mu = \int_{\mathbb{R}} f d\mu$$