

Integralit, U.I.

$$(E, \mathcal{E}, \mu), \mu(E) < \infty$$

Definition: $f \in L^1 \iff$ U.I. hvis $\forall \varepsilon > 0 \exists K > 0 : \sup_{f \in L^1} \int_E |f| d\mu \leq \varepsilon$
 $\{ |f| \geq K \}$

Lemma:

$$f \in L^1 \text{ U.I.} \iff \sup_{f \in L^1} \|f\|_1 < \infty, \forall \varepsilon > 0 \exists \delta > 0 \forall A \in \mathcal{E}, \mu(A) < \delta : \sup_{f \in L^1} \int_A |f| d\mu < \varepsilon$$

Setting: $(f_n), f \in M(E)$:

$$f_n \rightarrow f \text{ i } \mu\text{-mål}, (f_n) \text{ U.I.}$$

$$\iff f_n \rightarrow f \text{ i } L^1, ((f_n), f \in L^1)$$

Beris:

$\implies$

$$\exists \text{ delfølge } f_{n_k} \rightarrow f \text{ n.o.}$$

$$\int |f| d\mu \leq \liminf_{\text{Fatou}} \int |f_{n_k}| \leq \sup_n \|f_{n_k}\|_1 < \infty$$

Dvs. $f \in L^1$

$$|f - f_n| \leq |f| + |f_n| \implies |f - f_n| \text{ U.I. idet}$$

$$H_1, H_2 \text{ U.I.} \implies H_1 + H_2 \text{ U.I. (se Lemmaet)}$$

$$\int |f - f_n| d\mu \leq \int_{|f - f_n| \leq \varepsilon} |f - f_n| d\mu + \int_{\varepsilon \leq |f - f_n| < K} |f - f_n| d\mu + \int_{|f - f_n| \geq K} |f - f_n| d\mu$$

$$\leq \varepsilon \mu(E) + \mu(|f - f_n| \geq \varepsilon) K + \int_{|f - f_n| \geq K} |f - f_n| d\mu \quad 0 < \varepsilon < K$$

Vælg først ε lille K stor $\frac{\varepsilon}{2} + \mu(|f - f_n| \geq \varepsilon) K$ vælg derefter
 n stor

← :

$\|f - f_n\|_1 \xrightarrow{n \rightarrow \infty} 0$, dvs.

Lad $\varepsilon > 0$ være vilk.

Vælg $n_0 \geq 1 \quad \forall n \geq n_0: \|f - f_n\| \leq \varepsilon/2$

Da $\sum_{i=1}^{n_0} |f - f_i| \in L^1$ gælder $\exists \delta > 0 \quad \forall A \in \mathcal{E}, \mu(A) \leq \delta:$

$\int_A \sum_{i=1}^{n_0} |f - f_i| d\mu \leq \varepsilon/2$, dvs. $\sup_A \int_A |f - f_n| d\mu \leq \varepsilon$

Altså er $(|f - f_n|)$ U.I.

$|f_n| \leq |f - f_n| + |f|$ som for bm betyder dette (f_n) U.I.

$\forall \varepsilon > 0: \mu(|f - f_n| \geq \varepsilon) \stackrel{\text{Markov}}{\leq} \frac{1}{\varepsilon} \|f - f_n\|_1$, dvs. af følger konvergens i μ -mål. $\square$

Kriterier som giver U.I.

Dominant af en L^1 fkt.

Hvis $\exists p > 1 \quad \sup_{f \in \mathcal{F}} \|f\|_p < \infty$

Basis: $\int_{\{|f| \geq K\}} |f| d\mu \leq \int_{\{|f| \geq K\}} |f|^p |f|^{1-p} d\mu \leq \int |f|^p d\mu \cdot \sup_{x \geq K} x^{1-p} \quad \square$

(Scheffes Lemma hvis konv. n.o. eller i μ -mål, $f_n \rightarrow f$:
konv. i $L^1 \iff \|f_n\|_1 \rightarrow \|f\|_1$)

U.I. $\rightarrow$ optionalitet Optional Sampling.

Integraler, uligheder

Jensens Ulighed

$(\Omega, \mathcal{F}, P)$ ss-felt, $f \in L^1$ J-integral $P(f \in J) = 1$

$\forall \varphi: \mathbb{R} \rightarrow \mathbb{R}$ konvex på J og borelmålig:

$$\varphi(f) \in L(P), \quad m := \int f dP \in J, \quad \varphi(\int f dP) \leq \int \varphi(f) dP$$

Beris:

a, b endepkt. for J : $a \leq f \leq b$ n.o. $\Rightarrow a \leq m \leq b$

Hvis $m = a$: $a \in \mathbb{R}$ ($f \in L^1$) $P(f = a) = 1$ (ellers $m > a$)

$$\text{Dvs. } a \in J \quad \varphi(m) = \varphi(a) = \int \varphi(a) dP = \int \varphi(f) dP$$

Tilsvarende hvis $m = b$

Antag $m \in]a, b[$

J.L. teorien om konvekse fkt. $\exists k \in \mathbb{R}$

$$\forall x \in]a, b[: \quad \varphi(x) \geq k(x - m) + \varphi(m)$$

$$\Rightarrow \varphi(f) \geq k \underbrace{(f - m)}_{\geq 0} + \varphi(m) \text{ n.o.}$$

$$\Rightarrow \varphi(f) \in L \text{ og } \int \varphi(f) dP \geq \varphi(\int f dP)$$

□

$$\text{Korollar} \quad \|f\|_{p_1} \leq \|f\|_{p_2} \quad 0 < p_1 \leq p_2$$

Beris

$$t \mapsto |t|^{p_2/p_1} \text{ konvex på } \mathbb{R} \Rightarrow \left(\int |f|^{p_1} dP \right)^{p_2/p_1} \leq \int |f|^{p_2} dP$$

$$\Rightarrow \|f\|_{p_1} \leq \|f\|_{p_2} \leq \|f\|_{p_2} \xrightarrow{\text{monoton konvergens}} \|f\|_{p_1} \leq \|f\|_{p_2}$$

(Jensen Ulighed for Bm. benyttes $\varphi = \sup_n \varphi_n$ på J°)

Hölder's Ulikhed

$$\frac{1}{p} + \frac{1}{q} = 1 \quad p, q > 1, \quad f, g \in M(\mathcal{F})$$

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \quad (\text{specielt: } fg \in L^1 \text{ hvis } f \in L^p, g \in L^q)$$

Beweis:

Vi kan antage (påstanden trivialt ellers) at $f \in L^p, g \in L^q, P(f > 0)$:

Jf. monoton konvergens: $\nu(A) = \frac{1}{n} \int_A |f|^p dP$ er et ss-mål
 $\mu = \int_A |f|^p dP$

Sæt $u = |g \cdot f^{1-p}| 1_{\{f>0\}}$:

$$\begin{aligned} \frac{1}{n} \int |g f| dP &= \frac{1}{n} \int u |f|^p dP = \int u d\nu \\ &\stackrel{\text{Børllers}}{\leq} \left(\int u^q d\nu \right)^{1/q} = \left(\int |g|^q |f|^{q(1-p)} |f|^p dP / n \right)^{1/q} \\ &\stackrel{\text{Hölder}}{\leq} \left(\int |g|^q |f|^{q(1-p)} |f|^p dP / n \right)^{1/q} \\ &= n^{-\frac{1}{q}} \|g\|_q \implies \|fg\|_1 \leq \|f\|_p \|g\|_q \end{aligned}$$

□

Minkows ulighed: $1 \leq p < \infty$: $f, g \in M(\mathcal{F})$: $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

Markov: $P(|f| \geq r) \leq \frac{1}{r^p} \int |f|^p dP, \quad p > 0$

Beweis: $r 1_{\{|f| \geq r\}} \leq |f| \implies r^p 1_{\{|f| \geq r\}} \leq |f|^p \quad p > 0. \quad \square$

Doob's Ulikhed: $(X_n, \mathcal{F}_n)$ martingal: $p > 1$: $\|\sup_n X_n\|_p \leq \sup_n \|X_n\|_p$

Uafhængighed, σ -algebraer, s.v. og 0-1 love

$(\Omega, \mathcal{F}, P)$

T indexmngd.

Def. $(\mathcal{G}_t)_{t \in T} \subseteq \mathcal{F}$ uafh. hvis

$\mathcal{G}_{t_1}, \dots, \mathcal{G}_{t_n}$ uafh. $\forall t_1, \dots, t_n \in T, n \geq 1$, dvs.

$$\forall A_{t_1} \in \mathcal{G}_{t_1}, \dots, A_{t_n} \in \mathcal{G}_{t_n} : P(A_{t_1} \cap \dots \cap A_{t_n}) = \prod_{i=1}^n P(A_{t_i})$$

Pistand: $\mathcal{G}_1, \dots, \mathcal{G}_n \subseteq \mathcal{F}$ uafh. nf-stabile

$$\iff \sigma(\mathcal{G}_1), \dots, \sigma(\mathcal{G}_n) \text{ uafh.}$$

Bevis:

$\Leftarrow$: klar

$\Rightarrow$:

Set $\mathcal{G}_i^* = \mathcal{G}_i \cup \{\emptyset\}$, da er $\mathcal{G}_1^*, \dots, \mathcal{G}_n^*$ uafh. og nf stabile

Lad $G_2 \in \mathcal{G}_2^*, \dots, G_n \in \mathcal{G}_n^*$ være vilkårlig

def. $\mu_1(A) = P(A \cap G_2 \cap \dots \cap G_n)$

$$\nu_1(A) = P(A) \prod_{i=2}^n P(G_i), \quad A \in \mathcal{F}$$

$\mu_1|_{\mathcal{G}_1^*} = \nu_1|_{\mathcal{G}_1^*}$ og da $P \leq 1$ og P er σ -additivt er

μ_1, ν_1 endelige mål da $\mathcal{G}_1$ er (nf) stabil følger af

entydighedssættet for mål, at $\mu_1|_{\sigma(\mathcal{G}_1)} = \nu_1|_{\sigma(\mathcal{G}_1)}$

$$(\sigma(\mathcal{G}_1^*) = \sigma(\mathcal{G}_1))$$

Dvs. $\sigma(\mathcal{G}_1), \mathcal{G}_2^*, \dots, \mathcal{G}_n^*$ er uafh.

Lad $G_1 \in \sigma(g_1)$, $G_3 \in \mathcal{G}_3^*$, ..., $G_n \in \mathcal{G}_n^*$ være vilkårlige:

$$\text{def. } \mu_2(A) = P(G_1 \cap A \cap G_3 \cap \dots \cap G_n)$$

$$\nu_2(A) = P(A) P(G_1) \prod_{i=3}^n P(G_i) \quad A \in \mathcal{F}.$$

Som før ses $\sigma(g_1), \sigma(g_2), g_3, \dots, g_n \sim$ uafh.

Således fortsætter, dvs. $\sigma(g_1), \dots, \sigma(g_n) \sim$ uafh. $\square$

Dvs. $(g_t)_{t \in T}$ uafh. $(\mathcal{F})$ -stabil $\Leftrightarrow (\sigma(g_t))_{t \in T}$ uafh.

Lad $\mathcal{H}_s \subseteq (\mathcal{F}_t)_{t \in T}$, $s \in S$ være disjunkte mgdsystemer

Ved at betragte $\hat{\mathcal{H}}_s := \{A \mid A = G_1 \cap \dots \cap G_n, G_1 \in \mathcal{G}_{t_{s1}}^*, \dots, G_n \in \mathcal{G}_{t_{sn}}^*\}$

fås at det fremgikket idet $\sigma(\mathcal{H}_s) = \sigma(\hat{\mathcal{H}}_s)$ og

$\hat{\mathcal{H}}_s \sim \mathcal{F}$ -stabil at $(\sigma(\mathcal{H}_s))_{s \in S} \sim$ uafh.

Borel-Cantelli's lemmaer

$$1) P(\limsup A_n) = 0, \text{ hvis } \sum_{n=1}^{\infty} P(A_n) < \infty$$

$$2) P(\limsup A_n) = 1, \text{ hvis } \sum_{n=1}^{\infty} P(A_n) = \infty, (A_n) \text{ uafh.}$$

Kolmogorovs 0-1 lov

$$(X_n) \text{ uafh. s.v.}, \quad \mathcal{F}^\infty := \bigcap_{n=1}^{\infty} \sigma(X_j \mid j \geq n)$$

$$\text{Alle } A \in \mathcal{F}^\infty: P(A) = 0 \text{ el. } P(A) = 1$$

$$\text{Basis: } \mathcal{F}^\infty \perp \sigma(X_j \mid j \leq n), n \geq 1$$

$$\mathcal{F}_\infty = \bigcup_{n \geq 1} \sigma(X_j \mid j \leq n) \sim \mathcal{F}\text{-stabil}, \text{ dvs. } \mathcal{F}^\infty \perp \mathcal{F}_\infty$$

$$\Rightarrow \sigma(\mathcal{F}^\infty) \perp \sigma(\mathcal{F}_\infty) \quad (\mathcal{F}^\infty \mathcal{F}\text{-stabil}), \text{ men}$$

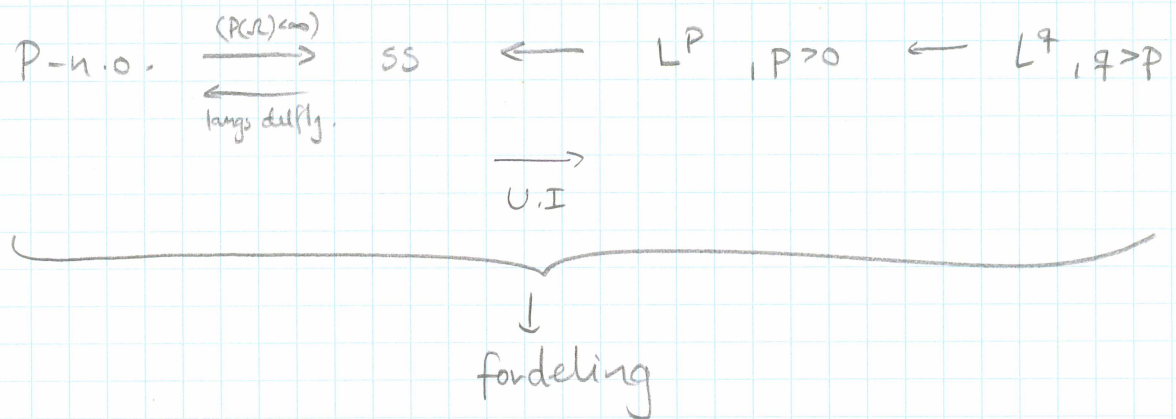
$$\mathcal{F}^\infty \subseteq \sigma(\mathcal{F}_\infty), \text{ dvs. } \forall F \in \mathcal{F}^\infty: P(F) = P(F)P(F) = P(F)^2$$

$$\Rightarrow P(F) = 0, \text{ eller } P(F) = 1 \quad \square$$

Konvergenzformen for s.v.

Konu. P-n.o., i ss., i L^p $p \geq 1$, i ford.

Sætning:



Beris:

P-n.o. $\rightarrow$ s.s.

$$\forall \varepsilon > 0 \quad P(\limsup_n \{|X - X_n| > \varepsilon\}) = 0$$

$$\mu(|X - X_n| > \varepsilon) \leq \mu\left(\bigcup_{k=n}^{\infty} \{|X - X_k| > \varepsilon\}\right) \downarrow 0, n \rightarrow \infty$$

s.s. $\rightarrow$ P-n.o. langs delfølge

$$\text{Vælg } (n_k) \quad P(|X - X_{n_k}| > 1/k) \leq 2^{-k}$$

$$\sum_{k=1}^{\infty} P(|X - X_{n_k}| > 1/k) < \infty \xRightarrow{\text{Borel-Cantelli}} P(\limsup_k \{|X - X_{n_k}| > 1/k\}) = 0$$

$$\Rightarrow P(\liminf_k \{|X - X_{n_k}| \leq 1/k\}) = 1$$

$$\Rightarrow \forall k \geq 1 \quad \forall l \geq k: |X - X_{n_l}| < 1/k \quad \text{n.o.}, \text{ des, } X_{n_k} \rightarrow X \text{ P-n.o.}$$

$$q > p, \quad L^q \rightarrow L^p$$

$f \in M(E)$: J. f. Jensen's Uighed

$$\left(\int (|f| h_n)^p dP \right)^{q/p} \leq \int (|f| h_n)^q dP$$

$\Downarrow$

$$\left(\int (|f| h_n)^p dP \right)^{1/p} \leq \|f\|_q$$

$\Downarrow$ monotone konvergens

$$\|f\|_p \leq \|f\|_q$$

$$L^p \rightarrow ss.$$

$$P(|X - X_n| > \varepsilon) \leq \frac{1}{\varepsilon^p} \|X - X_n\|_p^p$$

Markov

$$ss \rightarrow \text{ford.}$$

$$X_n \rightarrow X \text{ i ss} \xrightarrow{(+1)} f(X_n) \rightarrow f(X) \text{ i ss, } f \in BC(\mathbb{R})$$

$$\xrightarrow{(+2)} f(X_n) \rightarrow f(X) \text{ i } L^1, \text{ dvs. } E[f(X_n)] \rightarrow E[f(X)] = \int f d\mu$$

lille transform
sætning

hvilket pr. def. af konv. i fordeling betyder $X_n \Rightarrow X$

(+1)

Konv. i ss svarer til konv. mht. $d(X_n, X) = \int |X_n - X| dP$

som er en pseudometrik, da $|x+y| \leq |x| + |y|$

($\Rightarrow$ v.I. $\Leftarrow$ markov)

$\forall \varphi \in C(\mathbb{R}): \varphi(X_n) \rightarrow \varphi(X) \text{ i ss.}$

Antag modstridelse, at $\exists (n_k) \quad d(\varphi(X_{n_k}), \varphi(X)) > \varepsilon$

$\exists (n_{k_l}) : \varphi(X_{n_{k_l}}) \rightarrow \varphi(X) \text{ p.m.o.} \Rightarrow d(\varphi(X_{n_{k_l}}), \varphi(X)) < \varepsilon$

for et vist trin n_l (Lebesgue sæt. om dom. konv.)

(+2)

f begrænset $\Rightarrow (f(X_n))$ v.I.

Konv. i ss, L^p svarer til konv. mht. metrikk (pseudo)
ford. konv. — karakteristiske fkt.

Cauchy flg i ss- L^p er konvergent (L^p Banachrum)

Scheffes lemma: Hvis konv. i ss el. n.o.:

$$f_n \rightarrow f \text{ i } L^1 \iff \int f_n d\mu \rightarrow \int f d\mu$$

$$\left(\sum_{n=1}^{\infty} z_n \text{ konv. p.no.} \iff \sum_{n=1}^{\infty} z_n \text{ konv. i ss (} \underline{z} \text{) uafh.)} \right)$$

z_n uafh.

$$\sum_{n=1}^{\infty} z_n \text{ konv. i } L^2 \iff \sum_{n=1}^{\infty} \|z_n\|_2^2 < \infty$$

(Cauchy-flg + Pythagoras)

De Store Tals Lov

Kolmogorovs Store Tals Lov

(X_n) iid med middelværdi μ

$$\frac{1}{n} \sum_{i=1}^n X_i \rightarrow \mu \text{ P-n.o. og i } L^1$$

Lemma

(X_n) uafh. : $\sum_{n=1}^{\infty} X_n$ summabel P-n.o. $\iff$ konv. i ss

(specielt $L^p \implies$ P-n.o.)

Sætning: $(X_n) \in L^2$ uafh. :

$$\sum_{n=1}^{\infty} \text{Var}(X_n) / n^2 < \infty \implies \frac{1}{n} \sum_{i=1}^n (X_i - E[X_i]) \xrightarrow{n \rightarrow \infty} 0 \text{ n.o. } L^2$$

Basis:

Pr. uafh. og j.f. (*) har vi, at $\sum_{n=1}^{\infty} \frac{X_n - E[X_n]}{n}$ konv. i L^2

$\implies$
Kronechus lemma
+ lemma

$$\frac{1}{n} \sum_{i=1}^n X_i - E[X_i] \xrightarrow{n \rightarrow \infty} 0 \text{ P-n.o.}$$

$$\text{og } E\left[\left|\frac{1}{n} \sum_{i=1}^n X_i - E[X_i]\right|^2\right] = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) \xrightarrow{n \rightarrow \infty} 0 \text{ j.f. Kronecher}$$

□

Basis for Kolmogorovs Store Tals Lov:

$$X_n = X_n 1_{\{|X_n| \leq n\}} + X_n 1_{\{|X_n| > n\}}$$

$$\text{Var}(\bar{X}_n 1_{\{|X_n| \leq n\}}) \leq E[X_n^2; |X_n| \leq n]$$

$$\sum_{n=1}^{\infty} \text{Var}(\bar{X}_n 1_{\{|X_n| \leq n\}}) / n^2 \leq E[X_1^2] \sum_{n=1}^{\infty} \frac{1}{n^2} \leq C E[|X_1|] < \infty$$

monoton
konvergens

Ulynd
vs DGE 7
(Ovsum x^{-2})

Dvs. jf. ovenstående sætn., at

$$\frac{1}{n} \sum_{i=1}^n \bar{X}_i 1_{\{|X_i| \leq n\}} - E[\bar{X}_1 1_{\{|X_1| \leq n\}}] \rightarrow 0 \quad \text{P-n.o. og i } L^2$$

$$\bar{X}_1 1_{\{|X_1| \leq n\}} \uparrow \bar{X}_1 \Rightarrow \lim_{n \rightarrow \infty} E[\bar{X}_1 1_{\{|X_1| \leq n\}}] = \mu$$

Lebesgues
sætn. om
den konv.

$$\Rightarrow \frac{1}{n} \sum_{i=1}^n E[\bar{X}_i 1_{\{|X_i| \leq n\}}] \xrightarrow{n \rightarrow \infty} \mu$$

$$\sum_{n=1}^{\infty} P(\bar{X}_n 1_{\{|X_n| > n\}} \neq 0) = \sum_{n=1}^{\infty} P(|X_1| > n) < \infty \quad (X_1 \in L^1 + \text{int. funktion})$$

$$\text{Dvs. } P(\liminf \{|X_n| \leq n\}) = 1, \text{ dvs. } \lim_{n \rightarrow \infty} \bar{X}_n 1_{\{|X_n| > n\}} = 0 \quad \text{P-n.o.}$$

$$\text{Derved er } \frac{1}{n} \sum_{i=1}^n \bar{X}_i 1_{\{|X_i| > n\}} = 0 \quad \text{P-n.o.}$$

Alt i alt:

$$\frac{1}{n} \sum_{i=1}^n \bar{X}_i = \frac{1}{n} \sum_{i=1}^n \bar{X}_i 1_{\{|X_i| > n\}} + \frac{1}{n} \sum_{i=1}^n \bar{X}_i 1_{\{|X_i| \leq n\}} \xrightarrow{n \rightarrow \infty} \mu \quad \text{P-n.o.}$$

Pga. uafh. er $\{\bar{X}_n - E[\bar{X}_n]\}_{n \geq 1}$ U.I., dvs.

$\{\frac{1}{n} \sum_{i=1}^n \bar{X}_i - E[\frac{1}{n} \sum_{i=1}^n \bar{X}_i]\}_{n \geq 1}$ er U.I. og derved er

konvergens også i L^1 .

□

V.hj.a. Khinchins Ulyked

(Ej) Bernoulli, $\alpha > 0 \exists C_\alpha, C_\alpha$:

$$C_\alpha \left(\sum_{i=1}^n b_i^2 \right)^{\alpha/2} \leq E \left| \sum_{i=1}^n b_i \varepsilon_i \right|^\alpha \leq C_\alpha \left(\sum_{i=1}^n b_i^2 \right)^{\alpha/2} \quad \forall n \geq 1, (b_i) \subseteq$$

Symmetriseringsteknik $Z \sim Z'$: $Z - Z'$ symmetrisk kan man
vise

Maczkiewicz-Zygmunds Store Tals Lov

$$1 \leq q < 2, (I_n) \text{ iid} \in L^q: \frac{1}{n^{1/q}} \sum_{i=1}^n I_i - E[I_i] \rightarrow 0, \text{ no } L^q$$

L^α :

$$(X_n) \text{ uafh.} \in L^1$$

$$\alpha \leq 2: \sum_{n=1}^{\infty} E |X_n - E[X_n]|^\alpha / n^\alpha < \infty \Rightarrow \frac{1}{n} \sum_{i=1}^n X_i - E[X_i] \rightarrow 0 \text{ n.o. } L^1$$

$$\alpha > 2: \sum_{n=1}^{\infty} E |X_n - E[X_n]|^\alpha / n^{\alpha/2} < \infty \Rightarrow \frac{1}{n} \sum_{i=1}^n X_i - E[X_i] \rightarrow 0 \text{ n.o. } L^1$$

Derivere et resultat for uafhængige s.v.

Konvergens i $\mathbb{R}^n$, Kontinuitetssætning for karakteristiske fkt.

Sætning:

$(\underline{x}_k)$ s.v.

$$\underline{x}_k \Rightarrow \mu \implies \varphi_{\underline{x}_k}(t) \xrightarrow[k \rightarrow \infty]{} \hat{\mu}(t)$$

Hvis $\lim \varphi_{\underline{x}_k}(t) = \gamma(t)$ kont. i 0: $\underline{x}_k \Rightarrow \mu$, $\hat{\mu} = \gamma$

Lemma (Helly-Bray)

En strøm følge af s.v. har en delfolg. som konv. i ford.

En folg. af s.v. konv. i ford. hvis den er strøm og har et linesphæ.

Lemma: $(\underline{x}_k)$ strøm hvis

$$\forall \varepsilon > 0 \exists \eta > 0: \liminf E[e^{-\eta \|\underline{x}_k\|^2}] > 1 - \varepsilon$$

Basis for sætningen:

Jf. def. på konv. i ford: $\forall f \in bC(\mathbb{R}^n): E[f(\underline{x}_k)] \rightarrow \int f d\mu$

$x \mapsto \cos(t \cdot x)$, $\sin(t \cdot x) \in bC(\mathbb{R}^n)$ dermed er

$$\varphi_{\underline{x}_k}(t) \xrightarrow[k \rightarrow \infty]{} \hat{\mu}(t)$$

Hvis $(\underline{X}_n)$ er stram følger af Helly-Bray, at
 $\exists (k_\varepsilon) : \underline{X}_{k_\varepsilon}$ konv. i fordeling af det ovenstående
 ses at $\varphi_{\underline{X}_{k_\varepsilon}} \rightarrow \gamma$, og γ er den kar. fht. for
 givnefordelingen, des. $(\underline{X}_n)$ har et limespl. γ .

Altså er $(\underline{X}_n)$ konv. i fordeling $\underline{X}_n \rightarrow \mu$, $\hat{\mu} = \gamma$.

Vi ser, at $|\gamma| \leq 1$ og $\gamma(0) = 1$

$$\underline{U} \sim N(0, 2 \cdot I_n) \quad \varphi_{\underline{U}}(\underline{t}) = e^{-\|\underline{t}\|^2}$$

Vælg $a_n \rightarrow 0$ $\gamma(a_n \underline{U}) \xrightarrow{n \rightarrow \infty} 1$ idet γ er kont. i 0

Af Lebesgues Sætn. om domineret konvergens følger, at
 $\forall \varepsilon > 0 \exists n \geq 1$:

$$1 - \varepsilon < E[\gamma(a_n \underline{U})] = \lim_k E[\varphi_{\underline{X}_k}(a_n \underline{U})] = \lim_{k \rightarrow \infty} E[\varphi_{\underline{U}}(a_n \underline{X}_k)] \\ = \lim_{k \rightarrow \infty} E[e^{-a_n^2 \|\underline{X}_k\|^2}]$$

Dermed at $\liminf E[e^{-a_n^2 \|\underline{X}_k\|^2}] > 1 - \varepsilon$

Altså er $(\underline{X}_n)$ stram.

□

Korollar $\underline{X}_k \Rightarrow \underline{X} \parallel \underline{Y}_k \Rightarrow \underline{Y} : (\underline{X}_k, \underline{Y}_k) \Rightarrow (\underline{X}, \underline{Y})$

idet $\varphi_{\underline{X}}(\underline{t}) \varphi_{\underline{Y}}(\underline{t})$ er kont. i 0.

Den Centrale Grænseværdisætning

Lyapounovs Sætning

$\{X_{nj} \mid 1 \leq j \leq n\}_{n \geq 1}$ uafh. tilfældsskema i L^2

$$\mu_{nj} = E[X_{nj}], \quad \sigma_{nj}^2 = \text{Var}(X_{nj}), \quad s_n = \sqrt{\sigma_{n1}^2 + \dots + \sigma_{nn}^2}$$

Antag $s_n > 0 \quad \forall n \geq 1$

Hvis: $\exists \alpha > 2: \lim_{n \rightarrow \infty} \frac{1}{s_n^\alpha} \sum_{j=1}^n E|X_{nj} - \mu_{nj}|^\alpha = 0$

da gælder, at $U_n := \frac{1}{s_n} \sum_{j=1}^n X_{nj} - \mu_{nj} \Rightarrow N(0,1)$

Bm: Hvis $X_{nj} - \mu_{nj}$ er begrænset $\forall j, n$ er dette opfyldt hvis $s_n \rightarrow \infty$
(se p. i $\alpha=3 \quad \frac{1}{s_n} \rightarrow 0$)

Bevis:

$$Z_{nj} := X_{nj} - \mu_{nj}$$

$$\text{Def. } \mu_n: \mathcal{F} \times 2^{\{1, \dots, n\}} \quad \mu_n(A) = \sum_{j=1}^n E[1_A(\cdot, j) \mid \frac{Z_{nj}}{s_n}]^2$$

μ_n er et ss-mål

$\alpha > 3$:

$$\begin{aligned} \frac{1}{s_n^\alpha} \sum_{j=1}^n E|Z_{nj}|^\alpha &= \int |f| d\mu_n \leq \left(\int |f|^{\alpha-2} d\mu_n \right)^{\frac{1}{\alpha-2}} \\ &= \left(\frac{1}{s_n^\alpha} \sum_{j=1}^n E|Z_{nj}|^\alpha \right)^{\frac{1}{\alpha-2}}, \quad f(\cdot, j) = \left| \frac{Z_{nj}}{s_n} \right| \end{aligned}$$

Dvs. vi kan nøjes med at betragte $\alpha \in]2, 3]$

Lemma:

$$|\varphi_Z(t) - 1 - itE[Z] + \frac{t^2}{2}E[Z^2]| \leq |t|^\alpha E|Z|^\alpha$$

$$\alpha \in [2, 3], Z \in L^2$$

$$z_1, \dots, z_n, z'_1, \dots, z'_n \in \mathbb{C} \quad \left| \prod_{i=1}^n z_i - \prod_{i=1}^n z'_i \right| \leq \sum_{i=1}^n |z_i - z'_i|$$

$$|\log(1-x) + x| \leq 2x^2, |x| \leq \frac{1}{2}$$

Beweis für satzung 1

$$|\varphi_{Z_{nj}}(t) - 1 + \frac{t^2}{2} \frac{\sigma_{nj}^2}{s_n^2}| \leq |t|^\alpha E|Z_{nj}|^\alpha / s_n^\alpha$$

$$|\varphi_{U_n}(t) - \prod_{j=1}^n (1 - \frac{t^2}{2} \frac{\sigma_{nj}^2}{s_n^2})| \leq |t|^\alpha \sum_{j=1}^n E|Z_{nj}|^\alpha / s_n^\alpha \xrightarrow{n \rightarrow \infty} 0$$

$$\text{Dus: } \varphi_{U_n}(t) \rightarrow \lim_{n \rightarrow \infty} \prod_{j=1}^n (1 - \frac{t^2}{2} \frac{\sigma_{nj}^2}{s_n^2}) \quad (*)$$

$$\varphi_{N(0,1)}(t) = e^{-\frac{1}{2}t^2}$$

$$\left| \sum_{j=1}^n \log(1 - \frac{t^2}{2} \frac{\sigma_{nj}^2}{s_n^2}) + \frac{1}{2}t^2 \right| = \left| \sum_{j=1}^n (\log(1 - \frac{t^2}{2} \frac{\sigma_{nj}^2}{s_n^2}) + \frac{t^2}{2} \frac{\sigma_{nj}^2}{s_n^2}) \right|$$

$$\leq \sum_{j=1}^n 2 \frac{t^4}{4} \frac{\sigma_{nj}^4}{s_n^4} \leq \frac{t^4}{2} \max_{1 \leq j \leq n} \frac{\sigma_{nj}^2}{s_n^2} \sum_{j=1}^n \frac{\sigma_{nj}^2}{s_n^2} = \frac{t^4}{2} \max_{1 \leq j \leq n} \frac{\sigma_{nj}^2}{s_n^2} \xrightarrow{n \rightarrow \infty} 0$$

$$\text{idet } \max_{1 \leq j \leq n} \frac{\sigma_{nj}^2}{s_n^4} = \max_{1 \leq j \leq n} E \left| \frac{Z_{nj}}{s_n} \right|^2 \leq \left(\frac{1}{s_n^\alpha} \sum_{j=1}^n E|Z_{nj}|^\alpha \right)^{\frac{2}{\alpha}} \xrightarrow{n \rightarrow \infty} 0$$

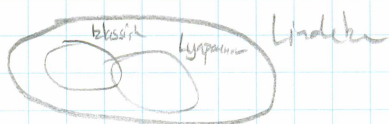
Jensen's
Ungleich
 $\varphi(x) = x^{\frac{\alpha}{2}}$

□

$$\text{Lindeberg: } \frac{1}{s_n^2} \sum_{j=1}^n E[Z_{nj}^2; \{|Z_{nj}| > \varepsilon s_n\}] \xrightarrow{n \rightarrow \infty} 0$$

Herunder specielt den klassiske udg. (X_n) iid:

$$\frac{1}{\sqrt{noz}} \sum_{j=1}^n X_j - \mu \xrightarrow{d} N(0,1)$$



Derveder en seer, liden momentel
men med momenta efter boel-transformation

husk formula $\left| \sum_{j=1}^n x_j \right|^q \leq n^{q-1} \sum_{j=1}^n |x_j|^q \quad q \geq 1$

Betingede middelværdier

$(\Omega, \mathcal{F}, P)$ ss-rum $\mathcal{U} \subseteq \mathcal{F}$ del σ -algebra

Proposition:

$\forall X \in L^1(P) \exists (! \text{ u.o.}) E[X|\mathcal{U}] \in L^1(P, \mathcal{U})$ (entydighed u.o. klar)

$$\forall A \in \mathcal{U}: \int_A X dP = \int_A E[X|\mathcal{U}] dP$$

Beweis for $X \in L^2(P)$:

$L^2(P, \mathcal{U})$ er et lukket underrum af $L^2(P)$

($f_n \rightarrow f$ i $L^2(P, \mathcal{U}) \Rightarrow f_n \rightarrow f$ u.o., dvs f $\mathcal{U}$ -målbar)

Dvs. jf. projektionssætningen $\exists X_{\mathcal{U}} \in L^2(P, \mathcal{U})$: $X - X_{\mathcal{U}} \in (L^2(P, \mathcal{U}))^{\perp}$

Da $1_A \in L^2(P, \mathcal{U})$, $A \in \mathcal{U}$: $\int_A X dP = \int_A X_{\mathcal{U}} dP \quad \square$

Lemma: $X, Y \in L^2(P)$: $(X - Y)_{\mathcal{U}} = X_{\mathcal{U}} - Y_{\mathcal{U}}$

$$E|X_{\mathcal{U}}| \leq E|X|$$

Beweis:

$$X - Y - (X_{\mathcal{U}} - Y_{\mathcal{U}}) \in (L^2(P, \mathcal{U}))^{\perp} \Rightarrow X_{\mathcal{U}} - Y_{\mathcal{U}} = (X - Y)_{\mathcal{U}}$$

$$E|X_{\mathcal{U}}| = \int_{\{X_{\mathcal{U}} \geq 0\}} X_{\mathcal{U}} dP - \int_{\{X_{\mathcal{U}} < 0\}} X_{\mathcal{U}} dP = \int_{\{X_{\mathcal{U}} \geq 0\}} X dP - \int_{\{X_{\mathcal{U}} < 0\}} X dP \leq E|X| \quad \square$$

Lad $X \in L^1(P)$

Sæt $X_n = X \cdot 1_{\{|X| \leq n\}}$ ($X_n \rightarrow X$ P-u.o. og i L^1 (Lebesgue))

$$\begin{aligned} & \text{Idet } \|E[X_n|\mathcal{U}] - E[X_m|\mathcal{U}]\|_1 \\ &= \|E[X_n - X_m|\mathcal{U}]\|_1 \leq \|X_n - X_m\|_1, \quad \exists X_{\mathcal{U}} \in L^1(P, \mathcal{U}) \end{aligned}$$

$E[X_n|\mathcal{U}] \rightarrow X_{\mathcal{U}}$ i L^1

($\exists n$ $E|X_n|\mathcal{U} \rightarrow X_{\mathcal{U}}$ u.o. $\Rightarrow X_{\mathcal{U}} \leftarrow \mathcal{U}$ -målbar)

$$\text{Idet } \forall A \in \mathcal{A}: |E[Z_n | A] - E[Z | A]|$$

$$= |E[Z_n - Z | A]| \leq E[|Z_n - Z| | A] \leq E|Z_n - Z| \xrightarrow{n \rightarrow \infty} 0$$

$$Z_n \rightarrow Z : L^1(P, \mathcal{A})$$

$$\forall A \in \mathcal{A}:$$

$$\int_A Z dP = \lim \int_A Z_n dP = \lim \int_A E[Z_n | \mathcal{A}] dP = \int_A Z dP$$

□

Eksempler på resultater:

$$\text{Hvis } X \in L^1(P), \varphi: \mathbb{R} \text{ kontinuert} \rightarrow \mathbb{R} \quad \varphi(X) \in L^1, P(X \in I) = 1 \\ \varphi(E[X | \mathcal{A}]) \leq E[\varphi(X) | \mathcal{A}]$$

Beweis:

$$\exists \varphi_n \text{ st\ae}t\text{t\ae}k\text{t. } \forall X \in I: \varphi(X) = \sup_n \varphi_n(X)$$

$$\varphi \geq \varphi_n \quad \forall n \quad E[\varphi(X) | \mathcal{A}] \geq \sup_n E[\varphi_n(X) | \mathcal{A}] = \sup_n E[\varphi_n(X) | \mathcal{A}] = \sup_n \varphi_n(E[X | \mathcal{A}]) = \varphi(E[X | \mathcal{A}])$$

$$(P(X \in I) = 1 \Rightarrow P(E[X | \mathcal{A}] \in I) = 1, P(X \in I) = 1 \Rightarrow P(E[X | \mathcal{A}] \in I) = 1$$

$$\text{Hvis } H: \mathbb{R}^2 \rightarrow \mathbb{R} \text{ b\ae}g\text{r\ae}n\text{set b\ae}nd\text{f\ae}h\text{t. } X \text{ u\ae}n\text{d\ae}b\text{g\ae}et \mathcal{F} \text{ u\ae}t \\ E[H(X, \mathcal{F}) | \mathcal{A}] = E[H(x, \mathcal{F})] \Big|_{x=X}$$

$$\text{Antag } H \text{ pos. (se at p\ae} \ H = H^+ - H^-)$$

$$H(x, \mathcal{F}) \text{ er m\ae}l\text{b\ae}l\text{ig og integrabel j\ae}f. \text{ Tonelli. } A \in \mathcal{A}:$$

$$\int_A H(X, \mathcal{F}) dP = \int_{\substack{z \in \mathbb{R} \\ z=1_A}} H(x, y) \cdot z dP_{(X, \mathcal{F})}(x, y, z) = \int H(x, y) \cdot z dP_{(X, \mathcal{F})}(x, y) \otimes dP_{\mathbb{R}}(z)$$

$$\stackrel{\text{Tonelli}}{=} \int E[H(x, \mathcal{F})] z dP_{(X, \mathcal{F})}(x, z) = \int_A E[H(x, \mathcal{F})] \Big|_{x=X} dP$$

Jensen $\rightarrow$ sub/super martingaler

Martingale, konvergensætninger

Martingalkonvergensætning:

$(X_n, \mathcal{F}_n)$ sub el. supermartingal:

Hvis $\sup_n \|X_n\|_1 < \infty \Rightarrow X \rightarrow X_\infty$ P-n.o. og $X_\infty \in L^1$

Lemma:

Lad $U_{r,s}^n$ betegne antal ophøjninger over $[r,s]$ i intervallet $[0,n]$ for (X_n) supermartingal

$$(s-r) E[U_{r,s}^n] \leq E[X_n^-] + |r|$$

Beris for sætningen:

Af monotøn konvergens følger $(s-r) E[\sup_n U_{r,s}^n] \leq \sup_n \|X_n\|_1 + |r|$

Dvs. $P(U_{r,s} = \infty) = 0$, $r, s \in \mathbb{Q}$

$$\Rightarrow P(\bigcup_{r,s \in \mathbb{Q}} \{U_{r,s} = \infty\}) = 0$$

Antag $\omega \in \mathcal{N}$: $\liminf X_n(\omega) < \limsup X_n(\omega)$

$\exists r, s \in \mathbb{Q}$: $\liminf X_n(\omega) < r < s < \limsup X_n(\omega)$

$\Rightarrow U_{r,s}(\omega) = \infty$, dvs. $\omega \in \bigcup_{r,s \in \mathbb{Q}} \{U_{r,s} = \infty\}$ og

Derved er $\liminf X_n = \limsup X_n$ n.o.

Dvs. $\exists X_\infty \in M(\mathcal{E})$: $X_n \rightarrow X_\infty$ n.o.

Idet $\lambda P(\max_{1 \leq i \leq n} |X_i| > \lambda) \leq 3 \max_{1 \leq i \leq n} \|X_i\|_1$ maximaltilfald

ses at $\sup_n \|X_n\|_1 < \infty \Rightarrow |X_\infty| < \infty$ n.o.

$$E|X_\infty| \leq \liminf_n E|X_n| < \infty$$

□

Korollar: Hvis (X_n) U.I. martingal: $X_n = E[X_\infty | \mathcal{F}_n]$

Beweis:

$$X_n \rightarrow X_\infty \text{ n.o. og i } L^1$$

$\Downarrow$

$$\|E[X_\infty | \mathcal{F}_n] - E[X_m | \mathcal{F}_n]\|_1 \leq \|X_\infty - X_m\|_1, \quad n \geq m \rightarrow 0$$

$$\text{Dvs. } X_n = E[X_m | \mathcal{F}_n] = E[X_\infty | \mathcal{F}_n] \quad m \geq n$$

Levys Sætning:

$$X \in L^1(P), (\mathcal{G}_n) \text{ filter: } E[X | \mathcal{G}_n] \rightarrow E[X | \mathcal{G}_\infty] \text{ n.o. i } L^1$$

$$\tau (\mathcal{G}_n) \text{ stopetid: } X_\tau = E[X | \mathcal{G}_\tau] \quad (X_n := E[X | \mathcal{F}_n])$$

Beweis:

$$(X_n) \text{ U.I. martingal: } X_n = E[X_\infty | \mathcal{G}_n]$$

$$\text{Dvs. } E[X | \mathcal{G}_n] = E[X_\infty | \mathcal{G}_n] \text{ n.o.}$$

$$\forall n \geq 1, B \in \mathcal{G}_n: \int_B X dP = \int_B X_\infty dP$$

Da $X_\infty \in \mathcal{G}_\infty = \sigma(\cup \mathcal{G}_n)$ måkelig gælder altså

$$(\cup \mathcal{G}_n \text{ algebra derved af-stabil}) \text{ at } X_\infty = E[X | \mathcal{G}_\infty]$$

Jvf. den strukturerede udg. af optional sampling

$$X_{\tau \wedge n} = E[X_n | \mathcal{G}_{\tau \wedge n}] = E[X | \mathcal{G}_{\tau \wedge n}]$$

$$\text{Da } \sigma(\cup \mathcal{G}_{\tau \wedge n}) = \mathcal{G}_\tau \text{ ses } X_{\tau \wedge n} \rightarrow E[X | \mathcal{G}_\tau] \text{ n.o.}$$

af martingalhav. sætn.: $X_{\tau \wedge n} \rightarrow X_\tau$ n.o. , dvs

$$E[X | \mathcal{G}_\tau] = X_\tau$$

□

Martingale, Optional Sampling

Sætning:

$(X_n, \mathcal{F}_n)$ martingal $\sigma \leq \tau$ stop tid τ optional

σ er optional og $E[X_0] = E[X_\sigma] = E[X_\tau]$ og

$$X_\sigma = E[X_\tau | \mathcal{F}_\sigma]$$

Hjælpsætninger:

Martingalkonvergenssætn.

(Z_n) sub- eller supermartingal: $\sup_n \|Z_n\|_1 < \infty$:

$Z_n \rightarrow Z_\infty$ (i R) n.o. og $Z_\infty \in L^1(P)$

Korollar: (Z_n) martingal v.I. $Z_n = E[Z_\infty | \mathcal{F}_n]$

Levys sætning (delvis)

$Z \in L^1(P)$ $Z_n = E[Z | \mathcal{G}_n]$; $Z_n \rightarrow E[Z | \mathcal{G}_\infty]$ n.o. L^1

σ stop tid mht $(\mathcal{G}_n)$, $Z_\sigma = E[Z | \mathcal{G}_\sigma]$

Optional Sampling (strøbet udg)

τ stop tid $(M_n, \mathcal{F}_n)$ martingal: $M_{n \wedge \tau}$ martingal

$0 \leq \sigma \leq \tau$ begrænsede stop tid:

$$E[M_0] = E[M_\sigma] = E[M_\tau], \quad M_\sigma = E[M_\tau | \mathcal{F}_\sigma]$$

Beweis for optional sampling:

$\sigma_{1n} \subseteq \tau_{1n}$ if den skabede udgave:

$X_{\sigma_{1n}} = E[X_{\tau_{1n}} | \mathcal{F}_{\sigma_{1n}}]$, dvs. σ er optional, idet $\mathcal{H}$ u.i. den σ -alg. $\Rightarrow \{E[X | \mathcal{H}_n] | X \in \mathcal{H}, n \geq 1\}$ u.i.

Def. Martingalkonvergenssætning. $X_{\sigma_{1n}} \xrightarrow{n.o.} X_\sigma \in \mathcal{L}^1$, $X_{\tau_{1n}} \xrightarrow{n.o.} X_\tau \in \mathcal{L}^1$

Pga. u.i. sign. korollarer:

$$X_{\sigma_{1n}} = E[X_\sigma | \mathcal{F}_n], \quad X_{\tau_{1n}} = E[X_\tau | \mathcal{F}_n]$$

Ifly den skabede udg. fås

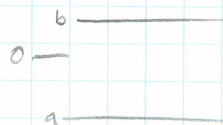
$$\begin{array}{ccc} X_{\sigma_{1n}} = E[X_{\tau_{1n}} | \mathcal{F}_{\sigma_{1n}}] = E[X_\tau | \mathcal{F}_{\sigma_{1n}}] & \xrightarrow[n.o.]{\text{levy}} & E[X_\tau | \mathcal{F}_\sigma] \\ \downarrow n.o. & & \\ X_\sigma & & \end{array}$$

$$\text{Dvs. } X_\sigma = E[X_\tau | \mathcal{F}_\sigma]$$

$$E[X_\sigma] = \underset{\text{skabede}}{E[X_\sigma]} = E[X_\tau] = E[X_\tau]$$

□

Anvendelse:



$$\tau = a \vee b$$

$$S_n^2 = n, \quad S_n$$

$$X_n = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$M[\tau] = ab, \quad P(X_\tau = -a) = \frac{b}{a+b}, \quad P(X_\tau = b) = \frac{a}{a+b}$$