become infinite. For this reason termwise differentiation is permissible, and we obtain

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) \tag{15}$$

except for an additive constant. If the terms corresponding to n and $-n$ are bracketed together, (15) can be written in the equivalent forms

$$\pi \cot \pi z = \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{z-n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \tag{16}$$

With this way of writing it becomes evident that both members of the equation are odd functions of z, and for this reason the integration constant must vanish. The equations (15) and (16) are thus correctly stated.

Let us now reverse the procedure and try to evaluate the analogous sum

$$\lim_{m \to \infty} \sum_{-m}^{m} \frac{(-1)^n}{z-n} = \frac{1}{z} + \sum_{1}^{\infty} (-1)^n \frac{2z}{z^2 - n^2} \tag{17}$$

which evidently represents a meromorphic function. It is very natural to separate the odd and even terms and write

$$\sum_{-(2k+1)}^{2k+1} \frac{(-1)^n}{z-n} = \sum_{n=-k}^{k} \frac{1}{z-2n} - \sum_{n=-k-1}^{k} \frac{1}{z-1-2n}.$$

By comparison with (16) we find that the limit is

$$\frac{\pi}{2} \cot \frac{\pi z}{2} - \frac{\pi}{2} \cot \frac{\pi(z-1)}{2} = \frac{\pi}{\sin \pi z},$$

and we have proved that

$$\frac{\pi}{\sin \pi z} = \lim_{m \to \infty} \sum_{-m}^{m} (-1)^n \frac{1}{z-n}. \tag{18}$$

EXERCISES

1. Comparing coefficients in the Laurent developments of $\cot \pi z$ and of its expression as a sum of partial fractions, find the values of

$$\sum_{1}^{\infty} \frac{1}{n^2}, \quad \sum_{1}^{\infty} \frac{1}{n^4}, \quad \sum_{1}^{\infty} \frac{1}{n^6}.$$

Give a complete justification of the steps that are needed.

2. Express

$$\sum_{-\infty}^{\infty} \frac{1}{z^3 - n^3}$$

in closed form.

3.2. Infinite Products. An infinite product of complex numbers

$$p_1 p_2 \cdots p_n \cdots = \prod_{n=1}^{\infty} p_n \tag{19}$$

is evaluated by taking the limit of the partial products $P_n = p_1 p_2 \cdots p_n$. It is said to converge to the value $P = \lim_{n \to \infty} P_n$ if this limit exists and i[s] different from zero. There are good reasons for excluding the value zero. For one thing, if the value $P = 0$ were permitted, any infinite produc[t] with one factor 0 would converge, and the convergence would not depen[d] on the whole sequence of factors. On the other hand, in certain con[-]nections this convention is too radical. In fact, we wish to express [a] function as an infinite product, and this must be possible even if th[e] function has zeros. For this reason we make the following agreement[:] The infinite product (19) is said to converge if and only if at most [a] finite number of the factors are zero, and if the partial products forme[d] by the nonvanishing factors tend to a finite limit which is different fro[m] zero.

In a convergent product the general factor p_n tends to 1; this is clea[r] by writing $p_n = P_n / P_{n-1}$, the zero factors being omitted. In view o[f] this fact it is preferable to write all infinite products in the form

$$\prod_{n=1}^{\infty} (1 + a_n) \tag{20}$$

so that $a_n \to 0$ is a necessary condition for convergence.

If no factor is zero, it is natural to compare the product (20) with th[e] infinite series

$$\sum_{n=1}^{\infty} \log (1 + a_n). \tag{21}$$

Since the a_n are complex we must agree on a definite branch of th[e] logarithms, and we decide to choose the principal branch in each ter[m]. Denote the partial sums of (21) by S_n. Then $P_n = e^{S_n}$, and if $S_n \to$ [S] it follows that P_n tends to the limit $P = e^S$ which is $\neq 0$. In oth[er] words, the convergence of (21) is a sufficient condition for the co[n-]vergence of (20).

In order to prove that the condition is also necessary, suppose that $P_n \to P \neq 0$, and choose a fixed value of $\log P$, for instance the value of the principal branch. With the corresponding value of $\arg P$ determine $\arg P_n$ by the condition $\arg P - \pi < \arg P_n \leqq \arg P + \pi$, and set $\log P_n = \log |P_n| + i \arg P_n$. We know that $S_n = \log P_n + h_n \cdot 2\pi i$, where h_n is a well-determined integer. For two consecutive terms we obtain

$$(h_{n+1} - h_n)2\pi i = \log (1 + a_{n+1}) + \log P_n - \log P_{n+1}.$$

We need pay attention only to the imaginary parts. As n is sufficiently large we have for instance $|\arg (1 + a_{n+1})| < 2\pi/3$, $|\arg P_n - \arg P| < 2\pi/3$, and $|\arg P_{n+1} - \arg P| < 2\pi/3$. These inequalities imply $|h_{n+1} - h_n| < 1$, and thus we conclude that $h_{n+1} = h_n$ for all sufficiently large n. Ultimately h_n is therefore constantly equal to a certain integer h, and we find that S_n tends to the limit $S = \log P + h \cdot 2\pi i$. We have proved:

Theorem 5. *The infinite product* $\prod_1^\infty (1 + a_n)$ *with* $1 + a_n \neq 0$ *converges simultaneously with the series* $\sum_1^\infty \log (1 + a_n)$ *whose terms represent the values of the principal branch of the logarithm.*

The question of convergence of a product can thus be reduced to the more familiar question concerning the convergence of a series. It can be further reduced by observing that the series (21) converges absolutely at the same time as the simpler series $\Sigma |a_n|$. This is an immediate consequence of the fact that

$$\lim_{z\to 0} \frac{\log (1 + z)}{z} = 1.$$

If either the series (21) or $\sum_1^\infty |a_n|$ converges, we have $a_n \to 0$, and for a given $\varepsilon > 0$ the double inequality

$$(1 - \varepsilon)|a_n| < |\log (1 + a_n)| < (1 + \varepsilon)|a_n|$$

will hold for all sufficiently large n. It follows immediately that the two series are in fact simultaneously absolutely convergent.

An infinite product is said to be absolutely convergent if and only if the corresponding series (21) converges absolutely. With this terminology we can state our result in the following terms:

Theorem 6. *A necessary and sufficient condition for the absolute convergence of the product* $\prod_1^\infty (1 + a_n)$ *is the convergence of the series* $\sum_1^\infty |a_n|$.

In the last theorem the emphasis is on absolute convergence. simple examples it can be shown that the convergence of $\sum_1^\infty a_n$ is ne sufficient nor necessary for the convergence of the product $\prod_1^\infty (1 +$

It is clear what to understand by a uniformly convergent in product whose factors are functions of a variable. The presen zeros may cause some slight difficulties which can usually be av by considering only sets on which at most a finite number of the fa can vanish. If these factors are omitted, it is sufficient to stud uniform convergence of the remaining product. Theorems 5 and 6 obvious counterparts for uniform convergence. If we examine the p we find that all estimates can be made uniform, and the conclusions to uniform convergence.

EXERCISES

1. Show that

$$\prod_{n=2}^\infty \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}.$$

2. Prove that for $|z| < 1$

$$\prod_{n=0}^\infty (1 + z^{2^n}) = \frac{1}{1 - z}.$$

3. Prove that

$$\prod_1^\infty \left(1 + \frac{z}{n}\right) e^{-z/n}$$

converges absolutely and uniformly on every compact set.

4. Prove that the value of an absolutely convergent product does not chang factors are reordered.

5. Show that the function

$$\theta(z) = \prod_1^\infty (1 + h^{2n-1}e^z)(1 + h^{2n-1}e^{-z})$$

where $|h| < 1$ is analytic in the whole plane and satisfies the functional equat

$$\theta(z + 2 \log h) = h^{-1}e^{-z}\, \theta(z).$$

3.3. Canonical Products. A function which is analytic in the plane is said to be *entire*, or *integral*. The simplest entire fun which are not polynomials are e^z, $\sin z$, and $\cos z$.

If $g(z)$ is an entire function, then $f(z) = e^{g(z)}$ is entire and $\neq 0$. versely, if $f(z)$ is any entire function which is never zero, let us that $f(z)$ is of the form $e^{g(z)}$. To this end we observe that the fu $f'(z)/f(z)$, being analytic in the whole plane, is the derivative of an function $g(z)$. From this fact we infer, by computation, that $f(z)e^{-g(z)}$ has the derivative zero, and hence $f(z)$ is a constant multiple of $e^{g(z)}$; the constant can be absorbed in $g(z)$.

By this method we can also find the most general entire function with a finite number of zeros. Assume that $f(z)$ has m zeros at the origin (m may be zero), and denote the other zeros by $a_1, a_2, \ldots, a_N$, multiple zeros being repeated. It is then plain that we can write

$$f(z) = z^m e^{g(z)} \prod_1^N \left(1 - \frac{z}{a_n}\right).$$

If there are infinitely many zeros, we can try to obtain a similar representation by means of an infinite product. The obvious generalization would be

$$f(z) = z^m e^{g(z)} \prod_1^\infty \left(1 - \frac{z}{a_n}\right). \tag{22}$$

This representation is valid if the infinite product converges uniformly on every compact set. In fact, if this is so the product represents an entire function with zeros at the same point (except for the origin) and with the same multiplicities as $f(z)$. It follows that the quotient can be written in the form $z^m e^{g(z)}$.

The product in (22) converges absolutely if and only if $\sum_1^\infty 1/|a_n|$ is convergent, and in this case the convergence is also uniform in every closed disk $|z| \leqq R$. It is only under this special condition that we can obtain a representation of the form (22).

In the general case convergence-producing factors must be introduced. We consider an arbitrary sequence of complex numbers $a_n \neq 0$ with $\lim_{n\to\infty} a_n = \infty$, and prove the existence of polynomials $p_n(z)$ such that

$$\prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{p_n(z)} \tag{23}$$

converges to an entire function. The product converges together with the series with the general term

$$r_n(z) = \log\left(1 - \frac{z}{a_n}\right) + p_n(z)$$

where the branch of the logarithm shall be chosen so that the imaginary part of $r_n(z)$ lies between $-\pi$ and π (inclusive).

For a given R we consider only the terms with $|a_n| > R$. In the disk $|z| \leqq R$ the principal branch of $\log(1 - z/a_n)$ can be developed in a Taylor series

$$\log\left(1 - \frac{z}{a_n}\right) = -\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \frac{1}{3}\left(\frac{z}{a_n}\right)^3 - \cdots.$$

We choose for $p_n(z)$ a partial sum of this series, ending with the term of degree m_n. Then $r_n(z)$ has the representation

$$r_n(z) = \frac{1}{m_n+1}\left(\frac{z}{a_n}\right)^{m_n+1} + \frac{1}{m_n+2}\left(\frac{z}{a_n}\right)^{m_n+2} + \cdots$$

and we obtain easily the estimate

$$|r_n(z)| \leqq \frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1}\left(1 - \frac{R}{|a_n|}\right)^{-1}. \tag{24}$$

Suppose now that the series

$$\sum_{n=1}^\infty \frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1} \tag{25}$$

converges. By the estimate (24) it follows first that $r_n(z) \to 0$, and hence $r_n(z)$ has an imaginary part between $-\pi$ and π as soon as n is sufficiently large. Moreover, the comparison shows that the series $\Sigma r_n(z)$ is absolutely and uniformly convergent for $|z| \leqq R$, and thus the product (23) represents an analytic function in $|z| < R$. For the sake of the reasoning we had to exclude the values $|a_n| \leqq R$, but it is clear that the uniform convergence of (23) is not affected when the corresponding factors are again taken into account.

It remains only to show that the series (25) can be made convergent for all R. But this is obvious, for if we take $m_n = n$, (25) becomes a power series with an infinite radius of convergence as seen either by use of the formula for the radius of convergence or by consideration of a majorant geometric series for any fixed value of R.

Theorem 7. *There exists an entire function with arbitrarily prescribed zeros a_n provided that, in the case of infinitely many zeros, $a_n \to \infty$. Every entire function with these and no other zeros can be written in the form*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^\infty \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}} \tag{26}$$

where the product is taken over all $a_n \neq 0$, the m_n are certain integers, and $g(z)$ is an entire function.

Uendelige produkter

$(p_n)_{n\geq 1} \subseteq \mathbb{C}$

$\prod_{n=1}^{\infty} p_n = p_1 \cdots p_n \cdots$

$P_n = \prod_{k=1}^{n} p_k$, $P_n' = \prod_{n=1}^{n}{}' p_k$ (alle 0-faktorer udeladt)

'1. Def. $\prod_{n=1}^{\infty} p_n$ er konvergent, hvis $\exists p \in \mathbb{C}: \lim_{n\to\infty} P_n = p \neq 0$ '

Def.: Det uendelige produkt $\prod_{n=1}^{\infty} p_n$ er konvergent, hvis
højst endeligt mange faktorer er 0, og
$\exists p \in \mathbb{C} : \lim_{n\to\infty} P_n' = p \neq 0$.

Antag, at $p_n \neq 0$ alle $n \in \mathbb{N}$

Bemærk: Hvis $P_n \xrightarrow[n\to\infty]{} P \neq 0$, så vil $p_n \xrightarrow[n\to\infty]{} 1$

da $p_n = \frac{p_1 \cdots p_n}{p_1 \cdots p_{n-1}} = \frac{P_n}{P_{n-1}} \xrightarrow[n\to\infty]{} \frac{P}{P} = 1$

Sætte $p_n = 1 + a_n$ $\prod_{n=1}^{\infty} p_n = \prod_{n=1}^{\infty} (1+a_n)$

En nødvendig betingelse for at $\prod_{n=1}^{\infty}(1+a_n)$ er konvergent, er at $a_n \xrightarrow[n\to\infty]{} 0$

$\prod_{n=1}^{\infty} (1+a_n) = \prod_{n=1}^{\infty} p_n = \lim_{n\to\infty} \prod_{k=1}^{n} p_k = P$

$\left(\log P = \lim_{n\to\infty} \log \prod_{k=1}^{n} p_k = \lim_{n\to\infty} \sum_{k=1}^{n} \log p_k = \lim_{n\to\infty} \sum_{k=1}^{n} \log(1+a_k) \Leftarrow \text{Undersøge}\right)$

Bm. $1+a_n = p_n \neq 0$

se på $\prod_{n=1}^{\infty}(1+a_n)$ og $\sum_{n=1}^{\infty} \log(1+a_n)$ vælger $\log(1+a_n) = \log|1+a_n| + i\arg(1+a_n)$ $\underset{>-\pi}{\overset{\leq \pi}{}}$

1° Antag først at $\prod_{n=1}^{\infty} \log(1+a_n)$ er konvergent

så er $\sum_{n=1}^{\infty}(1+a_n)$ konvergent

Bevis:

Givet $P_n = \prod_{k=1}^{n} (1+a_k) \xrightarrow[n\to\infty]{} P \neq 0$

$S_n := \sum_{k=1}^{n} \log(1+a_n)$ (afsnitssummen i $S = \sum \log(1+a_n)$)

$$P_n = e^{S_n} = e^{\sum^n \log(1+a_k)} = \prod_{k=1}^{n} e^{\log(1+a_n)} = \prod_{k=1}^{n} (1+a_n) \xrightarrow[n\to\infty]{} e^S$$

$$\sum^\infty \log(1+a_n) = S \Rightarrow \prod_{n=1}^{\infty} (1+a_n) = e^S$$

2° Antag at $P_n = \prod_{i=1}^{n} (1+a_k) \to P \neq$ $n\to\infty$

Så er $\sum^\infty \log(1+a_n)$ konvergent

Bevis: Vælg en fast værdi for $\log P$

$\log P = \log|P| + i \arg P$, $-\pi < \arg P \leq \pi$

Vælg $\arg P_n$ ved $\arg P - \pi < \arg P_n \leq \arg P + \pi$

$\log P_n = \log|P_n| + i \arg P_n$, $\arg P - \pi < \arg P_n \leq \arg P + \pi$

$S_n = \sum_{k=1}^{n} \log(1+a_k) = \log P_n + h_n 2\pi i$

$S_{n+1} = \sum_{k=1}^{n+1} \log(1+a_k) = \log P_{n+1} + h_{n+1} 2\pi i$

$\log(1+a_{n+1}) = S_{n+1} - S_n = \log P_{n+1} - \log P_n + (h_{n+1} - h_n) 2\pi i$

$(h_{n+1} - h_n) 2\pi i = \log(1+a_{n+1}) - \log P_{n+1} + \log P_n$

$= \log|1+a_n| - \log|P_{n+1}| + \log|P_n| + i\{\arg(1+a_{n+1}) - \arg P_{n+1} + \arg P_n\}$

1) $\log|1+a_{n+1}| - \log|P_{n+1}| + \log|P_n| = 0 \quad \forall n \geq N$, for et $N \geq 1$

2) $2\pi(h_{n+1}-h_n) = \arg(1+a_{n+1}) - \arg P_{n+1} + \arg P_n = 0 \quad \forall n \geq N$, for et $N \geq 1$

Bevis:

1)

$$0 \leftarrow \log|1+a_n| = \log|P_{n+1}| - \log|P_n| = \log\frac{|P_{n+1}|}{|P_n|} \longrightarrow 0 \quad n \to \infty$$

2)

$-\pi < \arg(1+a_n) \leq \pi$, $\arg P - \pi < \arg P_n \leq \arg P + \pi$, $P_n \xrightarrow[n\to\infty]{} P$

$\arg P - \pi < \arg P_{n+1} \leq \arg P + \pi$

$|\arg P_n| =$

$\exists N \geq 1 \ \forall n \geq N$: $|\arg(1+a_{n+1})| < \frac{2\pi}{3}$ (idet $1+a_{n+1} \xrightarrow[n\to\infty]{} 1$)

og $|\arg P_n - \arg P| < \frac{2\pi}{3}$ (idet $\arg P - \pi < \arg P_n \leq \arg P + \pi$)

og $|\arg P_{n+1} - \arg P| < \frac{2\pi}{3}$ (idet $\arg P - \pi < \arg P_{n+1} \leq \arg P + \pi$)

$$0 \leq |\arg(1+a_{n+1}) + \arg P_{n+1} + \arg P_n|$$

$$= |\arg(1+a_{n+1}) - (\arg P_{n+1} - \arg P) + (\arg P_n - \arg P)|$$

$$\leq |\arg(1+a_n)| + |\arg P_{n+1} - \arg P| + |\arg P_n - \arg P| < 2\pi$$

Dvs. $h_{n+1} - h_n = 0$, $n \geq N$ altså $h_n = h$, $n \geq N$. (hvor $h \in \mathbb{Z}$)

$S_n \to \log P + h \cdot 2\pi i$ fast h fra et visst trin $N \geq 1$

$n \to \infty$: $P_n \to P$ og $\arg P_n \to \arg P \Rightarrow \log P_n \to \log P$

Sætning: $\prod_{n=1}^{\infty} (1+a_n)$ er konvergent $\iff$ $\sum_{n=1}^{\infty} \log(1+a_n)$ konvergent med sum S

$(1+a_n) \neq 0$

med produkt P hvor $-\pi < \arg(1+a_n) \leq \pi$

$$\text{hvor } P = e^S, \quad S = \sum_{n=1}^{\infty} \log(1+a_n) + h \cdot 2\pi i$$

Sætning 2

$\sum_{n=1}^{\infty} \log(1+a_n)$ er abs. konvergent $\iff \sum_{k=1}^{\infty} |a_n| < \infty$

Bevis:

$$\frac{\log(1+z)}{z} \xrightarrow[z \to 0]{} 1 \qquad -\pi < \arg(1+z) \leq \pi$$

$$\frac{\log(1+z) - \log 1}{z} = \frac{d}{dz} \log(1+z)\Big|_{z=0} = 1$$

$$1-\varepsilon < \left| \frac{\log(1+a_n)}{a_n} \right| < 1+\varepsilon \quad , \quad n > N$$

$\Downarrow$

$$(1-\varepsilon)|a_n| < |\log(1+a_n)| < (1+\varepsilon)|a_n|$$

Dvs. $\sum_{n=1}^{\infty} |a_n| < \infty \implies \sum_{n=1}^{\infty} |\log(1+a_n)| < (1+\varepsilon) \sum_{n=1}^{\infty} |a_n| < \infty$

Vi bruger her 'Majoritetssætningen'

omvendt:

$$\sum_{n=1}^{\infty} |\log(1+a_n)| < \infty \implies |a_n| < \frac{1}{1-\varepsilon} |\log(1+a_n)| \implies \sum_{n=1}^{\infty} |a_n| < \infty$$

Opg.

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} \qquad |z|<1$$

$$-\frac{1}{z-1} = \frac{1}{1-z} = 1+z+z^2+\dots+z^n+\dots$$

$$\frac{1}{z-2} = \frac{\frac{1}{2}}{\frac{z}{2}-1} = -\frac{1}{2}\,\frac{1}{1-z/2} \qquad |z|<2 : \left|\frac{z}{2}\right|<1$$

$$= -\frac{1}{2}\left(1+\frac{z}{2}+\dots+\left(\frac{z}{2}\right)^n+\dots\right)$$

$|z|<1$

$$\frac{1}{(z-1)(z-2)} = -\frac{1}{2}\left\{1+\frac{z}{2}+\dots+\left(\frac{z}{2}\right)^n+\right\} + 1+z^2+\dots+z^n+\dots$$

$1<|z|<2$

$$\frac{1}{1-z} = \frac{1}{z}\left\{\frac{1}{\frac{1}{z}-1}\right\} = -\frac{1}{z}\left\{1+\frac{1}{z}+\dots+\frac{1}{z^n}+\dots\right\}$$

$$f(z) = -\frac{1}{z-1}+\frac{1}{z-2} = -\frac{1}{z}\left\{1+\frac{1}{z}+\dots+\frac{1}{z^n}+\dots\right\}$$

$$-\frac{1}{2}\left\{1+\frac{z}{2}+\dots+\left(\frac{z}{2}\right)^n+\dots\right\} = -\frac{1}{z^n}\dots-\frac{1}{z^2}-\frac{1}{z}$$

$$-\frac{1}{2}-\frac{1}{2^2}z\dots$$

Forelæsn. tirsdag 11.12.

$$\prod_{n=1}^{\infty} z_n = \prod_{n=1}^{\infty} (1+a_n) \qquad a_n \to 0 \quad n \to \infty$$

Følge $f_1(z), \ldots, f_n(z), \ldots \quad , z \in \Omega \subseteq \mathbb{C}$

$f_n(z)$ analytisk i Ω for alle $n \geq 1$

Antag $\forall z \in \Omega$: $\prod_{n=1}^{\infty} f_n(z)$ konvergent

dvs. $\forall z \in \Omega$ er $f_n(z) = 0$ for højst endelig mange $n = n_i$, $i = 1, \ldots, k$

og $\prod_{\substack{n \neq n_i \\ i=1,\ldots,k}} f_n(z)$ konvergent, dvs $\prod_{\substack{n=1 \\ n \neq n_i}}^{N} f_n(z) \to \prod_{\substack{n=1 \\ n \neq n_i}}^{\infty} f_n(z) \neq 0$

Uniform konvergens

Def. Lad $f(z) = \prod_{n=1}^{\infty} f_n(z)$ være punktvis konvergent i Ω

Så siges produktet at være uniformt konvergent i Ω, hvis

1) Der findes endeligt mange n, $n = n_1, \ldots, n_k$ så at $f_n(z) = 0$ for et $z \in \Omega$

2) $\prod_{n=1}^{\infty}{}' f_n(z) = \prod_{\substack{n=1 \\ n \neq n_1, \ldots, n_k}} f_n(z)$ uniformt konvergent produkt i Ω, dvs

$\prod_{\substack{n=1 \\ n \neq n_1, \ldots, n_k}}^{N} f_n(z) \longrightarrow \prod_{\substack{n=1 \\ n \neq n_1, \ldots, n_k}}^{\infty} f_n(z) \neq 0$ uniformt i Ω

altså $\forall \varepsilon > 0 \ \exists N \geq 1$ så at $\left| \prod_{\substack{n=1 \\ n \neq n_1, \ldots, n_k}}^{\infty} f_n(z) - \prod_{\substack{n=1 \\ n \neq n_1, \ldots, n_k}}^{N} f_n(z) \right| < \varepsilon$ for $n > N$

Vi minder om at hvis

$g_n(z)$ analytisk på Ω og $g_n(z) \underset{n \to \infty}{\Longrightarrow} g(z)$ uniformt på Ω,

så er $g(z)$ analytisk på Ω.

Altså er $\prod_{n=1}^{\infty}{}' f_n(z)$ analytisk $\neq 0$ i Ω

Så er $\prod_{n=1}^{\infty} f_n(z) = f_{n_1}(z) \cdots f_{n_k}(z) \cdot \prod_{n=1}^{\infty}{}' f_n(z)$ analytisk i Ω

Under uniform konvergens gælder følg. sætninger

Lemma: En nødvendig betingelse for at $f(z) = \prod_{n=1}^{\infty} f_n(z)$ er uniformt konvergent, er at $f_n(z) \underset{n\to\infty}{\longrightarrow} 1$ uniformt for $z \in \Omega$

$f_n(z) = 1 + a_n(z)$, $a_n(z) \underset{n\to\infty}{\longrightarrow} 0$ uniformt $z \in \Omega$

Bevis:

$$\left.\begin{array}{l} \prod_{n=1}^{N}{}' f_n(z) \underset{N\to\infty}{\longrightarrow} f(z) \neq 0 \quad \text{uniformt} \\ \prod_{n=1}^{N+1}{}' f_n(z) \underset{N\to\infty}{\longrightarrow} f(z) \neq 0 \quad \text{uniformt} \end{array}\right\} \Longrightarrow f_n(z) \underset{n\to\infty}{\longrightarrow} 1 \quad \text{uniformt}$$

Sætning 5': Det uendelige produkt $\prod_{n=1}^{\infty} (1 + a_n(z)) = f(z)$ med uniform konvergens hviss $\sum_{n=1}^{\infty} \log(1 + a_n(z))$ er uniformt konvergent hvor $\log(1 + a_n(z))$ er hovedværdien af log, dvs $\arg(1 + a_n(z)) \in]-\pi, \pi]$

I beviset for sætn. 5 kan $\varepsilon(N)$ vælges uafh. af $z \in \Omega$ under benyttelse af $\left| \prod_{n=1}^{\infty}{}' f_n(z) - \prod_{n=1}^{N}{}' f_n(z) \right| < \varepsilon$ for $n > N(\varepsilon)$ uafh. af $z \in \Omega$ (*)

Sætning 6' Det uendelige produkt $\prod_{n=1}^{\infty} (1 + a_n(z))$ er absolut og uniformt konvergent for $z \in \Omega$ hviss $\sum_{n=1}^{\infty} |a_n(z)|$ er uniformt konvergent for $z \in \Omega$

I beviset for sætn. 6 benyttes (*)

Kanoniske produkter af hele analytiske funktioner

Lad $f_n(z)$ være en hel analytisk funktion, dvs. analytisk i $\mathbb{C}$

Studere $\prod_{n=1}^{\infty} f_n(z)$

Ideen Hvis f er en hel analytisk fkt. med nulpkt. er

$z_1, \dots, z_n, \dots$ søger vi at skrive $f(z) = g(z) z^m (1-\frac{z}{z_1})(1-\frac{z}{z_2}) \dots$

$\dots (1-\frac{z}{z_k})$

(Husk der kan kun være tælleligt mange nulpunkter)

Lad g være en hel analytisk funktion.

Så er $e^{g(z)}$ en hel analytisk funktion og $e^{g(z)} \neq 0 \quad \forall z \in \mathbb{C}$

Omvendt:

Lemma Lad $f(z)$ være en hel analytisk fkt. $f(z) \neq 0 \quad \forall z \in \mathbb{C}$

Så er $f(z)$ af formen $f(z) = e^{g(z)}$, hvor $g(z)$ er en hel analytisk fkt.

Bevis:

$\frac{f'(z)}{f(z)}$ er en hel analytisk fkt.

Så er $\frac{f'(z)}{f(z)} = g'(z)$, hvor $g(z) = \int_0^z \frac{f'(t)}{f(t)} dt$

dvs. $f'(z) = f(z) g'(z)$

$$\frac{d}{dz}\{f(z) e^{-g(z)}\} = \{f'(z) - f(z) g'(z)\} e^{-g(z)} = 0 \implies f(z) e^{-g(z)} = c \in \mathbb{C}$$

$$\begin{aligned} f(z) &= c \cdot e^{g(z)} \\ &= e^{\log c} e^{g(z)} \\ &= e^{g(z) + \log c} \\ &= e^{g_1(z)}, \text{ hvor } g_1 \text{ hel analytisk} \end{aligned}$$

Antag at f er en hel analytisk fkt. med endeligt mange nulpkter.

$0, a_1, a_2, \dots, a_n$ $\quad a_i \neq 0$ og gentages med multiplicitet

Så er $f(z)$ af formen $f(z) = z^m \prod_{k=1}^{n} (1 - \frac{z}{a_k}) \underbrace{e^{g(z)}}_{h(z)}$, g hel analytisk

Bevis følger

Opgave Vis at $\int_0^\infty \frac{\log x}{1+x^2} dx = 0$ og $\int_0^\infty \frac{(\log x)^2}{1+x^2} dx = \frac{\pi^3}{8}$

1) $\int_0^\infty \frac{\log x}{1+x^2} dx$

$\log z = \log|z| + i \arg_{-\pi/2} z$

analytisk i $\mathbb{C} \setminus \{-i[0,\infty)\}$

$x > 0 \quad \log(-x) = \log(x e^{i\pi}) = \log x + i\pi$

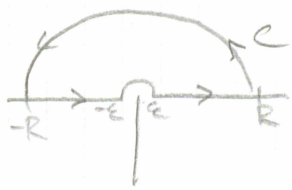
$\int_{-\infty}^0 \frac{\log x}{1+x^2} dx = \int_{-\infty}^0 \frac{\log(-x)}{1+x^2} dx$ $\quad \left(\int_0^\infty \frac{\log x}{1+x^2} dx = C \right.$ (konvergent reelt integral)

$= \int_{-\infty}^0 \frac{\log|x| + i\pi}{1+x^2} dx = \int_{-\infty}^0 \frac{\log(-x)}{1+x^2} dx + i\pi \int_{-\infty}^0 \frac{1}{1+x^2} dx$

$= -\int_0^\infty \frac{\log x}{1+x^2} + i\pi \int_{-\infty}^0 \frac{1}{1+x^2} dx = -C + i\pi \int_{-\infty}^0 \frac{1}{1+x^2} dx$

$\int_0^\infty \frac{\log x}{1+x^2} dx = 0$

2) $\int_0^\infty \frac{(\log x)^2}{1+x^2} dx = \frac{\pi^3}{8}$



$f(z) = \frac{(\log z)^2}{1+z^2}$

$\log z$ analytisk i en åben mængde som indeholder C

$$f(z) = \frac{(\log z)^2}{1+z^2} = \frac{(\log z)^2}{(z-i)(z+i)} \quad \text{simple pole i } z = \pm i$$

$$\int_C f(z)\,dz = 2\pi i \operatorname{res}(f, i) = 2\pi i \quad = 2\pi i\left(\frac{-\pi^2}{8i}\right) = -\frac{\pi^3}{4} \quad \text{, da}$$

$$\operatorname{res}(f, i) = \lim_{z \to i} \frac{(\log z)^2}{z+i} = \frac{(\log i)^2}{2i} = \frac{\left(\log 1 + i\frac{\pi}{2}\right)^2}{2i}$$

$$= \frac{\left(i\frac{\pi}{2}\right)^2}{2i} = \frac{-\pi^2/4}{2i} = -\frac{\pi^2}{8i}$$

$$\text{vis} \quad \int_0^\infty \frac{(\log x)^2}{1+x^2} = \frac{\pi^3}{8}$$

$$\int_C f(z)\,dz = \int_{-R}^{-\varepsilon} \frac{(\log|x| + i\pi)^2}{1+x^2}\,dx - \underbrace{\int_0^\pi \frac{(\log(\varepsilon e^{i\varphi}))^2}{1+(\varepsilon e^{i\varphi})^2}\,\varepsilon i e^{i\varphi}\,d\varphi}_{(z=\varepsilon e^{i\varphi},\ \varphi \in [0,\pi])}$$

$$+ \int_\varepsilon^R \frac{(\log x)^2}{1+x^2}\,dx + \underbrace{\int_0^\pi \frac{(\log Re^{i\varphi})^2}{1+(Re^{i\varphi})^2}\,Rie^{i\varphi}\,d\varphi}_{(z=Re^{i\varphi},\ \varphi\in[0,\pi])}$$

$$\left|\int_0^\pi \frac{(\log \varepsilon e^{i\varphi})^2}{1+(\varepsilon e^{i\varphi})^2}\,\varepsilon i e^{i\varphi}\,d\varphi\right| = \varepsilon\left|\int_0^\pi \frac{(\log\varepsilon + i\varphi)^2}{1+\varepsilon^2 e^{2i\varphi}}\,e^{i\varphi}\,d\varphi\right|$$

$$= \left|\int_0^\pi \frac{(\log\varepsilon)^2 - \varphi^2 + 2i\varphi\log\varepsilon}{1+\varepsilon^2 e^{2i\varphi}}\,e^{i\varphi}\,d\varphi\right|$$

$$\leq \int_0^\pi \varepsilon\,\frac{(\log\varepsilon)^2 + \varphi^2 + 2|\varphi|\,\log\varepsilon}{3/4} \qquad 1-\varepsilon^2 \geq 3/4 \Rightarrow \varepsilon < 1/2$$

$$= \frac{4}{3}\left(\varepsilon(\log\varepsilon)^2 + \varepsilon\pi^2 + 2\pi\varepsilon\log\varepsilon\right) \xrightarrow[\varepsilon\downarrow 0]{} 0$$

$$\left| \int_0^\pi \frac{(\log(Re^{i\varphi}))^2}{1+R^2e^{2i\varphi}} Re^{i\varphi} i\, d\varphi \right|$$

$$= R \left| \int_0^\pi \frac{(\log R + i\varphi)^2}{1+R^2e^{2i\varphi}} e^{i\varphi} d\varphi \right| \qquad 1+R^2e^{2i\varphi} \geq R^2-1 \qquad R \geq 2$$

$$\leq R \int_0^\pi \left| \frac{(\log R)^2 + \pi^2 + 2\pi \log R}{R^2-1} \right| d\varphi$$

$$\leq R \left\{ \frac{(\log R^2)}{R^2-1} + \frac{\pi^2}{R^2-1} + 2\pi \frac{\log R}{R^2-1} \right\} \xrightarrow[R\to\infty]{} 0$$

$$\int_{-R}^{-\varepsilon} \frac{(\log|x| + i\pi)^2}{1+x^2} dx = \int_{-R}^{-\varepsilon} \frac{(\log|x|)^2 - \pi^2 + 2i\pi \log|x|}{1+x^2} dx = (+)$$

$$\left(\int_\varepsilon^R \frac{(\log x)^2}{1+x^2} dx \xrightarrow[\substack{\varepsilon\to 0 \\ R\to\infty}]{} \int_0^\infty \frac{(\log x)^2}{1+x^2} dx \right)$$

$$(+) \to \int_{-\infty}^0 \frac{(\log(-x))^2 - \pi^2 + 2i\pi \log(-x)}{1+x^2} dx$$

$$\int_{-\infty}^0 \frac{(\log(-x))^2}{1+x^2} dx = \int_{-\infty}^0 \frac{(\log t)^2}{1+t^2} (-1)\, dt = \int_0^\infty \frac{(\log x)^2}{1+x^2} dx$$

$$\int_{-\infty}^0 \frac{-\pi^2}{1+x^2} dx = \int_\infty^0 \frac{-\pi^2(-1)\, dt}{1+t^2} = -\pi^2 \int_0^\infty \frac{dx}{1+x^2} = -\pi^2 [\operatorname{Arctan} x]_0^\infty$$

$$= -\pi^2 \frac{\pi}{2} = -\frac{\pi^3}{2}$$

$$\int_{-\infty}^0 \frac{2i\pi \log(-x)}{1+x^2} dx = \int_0^\infty \frac{2i\pi \log x}{1+x^2} dx = 0$$

$$-\frac{\pi^3}{4} = 2\int_0^\infty \frac{(\log x)^2}{1+x^2} dx - \frac{\pi^3}{2} \implies \int_0^\infty \frac{(\log x)^2}{1+x^2} dx = \frac{\pi^3}{8}$$

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Kanoniske produkter

Hvis $f(z)$ er hel analytisk, $f(z) \neq 0 \ \forall z$

så findes $g(z)$ hel analytisk så $f(z) = e^{g(z)}$

Antag nu at $f(z)$ er hel analytisk og $f(z)$ har endeligt mange nulpunkter, $a_0 = 0$, orden $m \geq 0$, $a_2, a_3, \ldots, a_n \neq 0$

$|a_1| \leq |a_2| \leq |a_3| \leq \ldots \leq |a_n|$

$\begin{cases} a_1 = a_2 = \ldots = a_{k_1} & a_1 \text{ nulpkt. af mult. } k_1 \\ a_{k_1+1} = \ldots = a_{k_1+k_2} & a_2 \text{ nulpkt af mult. } k_2 \end{cases}$

p forskellige rødder $\neq 0$

Så gælder at

$$f(z) = z^m \prod_{i=1}^{n} (1 - z/a_i)\, h(z) \quad , \quad h(z) \neq 0 \ \forall z$$

$$= z^m \prod_{i=1}^{n} (1 - z/a_i)\, e^{g(z)} \quad , \quad g(z) \text{ hel}$$

Bevis:

antag at $m > 0$

Så er for z nær 0

$f(z) = b_m z^m + b_{m+1} z^{m+1} + \ldots$

$f_1(z) = \dfrac{f(z)}{z^m} = b_m + b_{m+1} z + \ldots$

f_1 hævelig singularitet i $0 \rightarrow \tilde{f}_1(z) = \begin{cases} f_1(z), & |z| < \varepsilon \\ b_m, & z = 0 \end{cases}$

$f_1(z) = \dfrac{f(z)}{z^m}$ analytisk for $|z| < \varepsilon$

udvikle $f_1(z)$ om $z = a_1$ $\quad f_1(z) = c_{k_1}(z - a_1)^{k_1} + c_{k_1+1}(z - a_1)^{k_1+1} + \ldots$ $\quad c_{k_1} \neq 0$

$h(z) = \dfrac{f_1(z)}{(z - a_1)^{k_1}} = c_{k_1} + c_{k_1+1}(z - a_1) + \ldots \Rightarrow \tilde{f}_2(z) = \begin{cases} h(z), & \varepsilon > |z - a_1| > 0 \\ c_{k_1}, & z = a_1 \end{cases}$

$$h(z) = \frac{f(z)}{z^m (z-a_1)^{k_1}} = \frac{f(z)}{z^m (-a_1)^{k_1} (1-\frac{z}{a_1})^{k_1}}$$

For en funktion

$$\hat{f}(z) = \frac{f(z)}{z^m (1-\frac{z}{a_1})^{k_1} \cdots (1 - z/a_p)^{k_p} (-a_1)^{k_1} \cdots (-a_p)^{k_p}}$$

$$= \frac{f(z)}{z^m (1-z/a_1) \cdots (1-z/a_p) c} \neq 0 \quad \forall z$$

D.v.s. $f(z) = z^m (1 - z/a_1) \cdots (1-z/a_p)\, h(z) \qquad h(z) \neq 0 \ \forall z$

$$= z^m (1 - z/a_1) \cdots (1-z/a_p)\, e^{g(z)}$$

Sætning. Lad $f(z)$ hel analytisk og have uendeligt mange nulpkt. er $0, a_1, \ldots, a_n, \ldots \quad 0 < |a_1| \leq |a_2| \leq \ldots$

(dvs. indenfor en hvis model: endeligt mange nulpkt.)

Hvis $\prod_{n=1}^{\infty} (1 - z/a_n)$ er uniformt konvergent

på $\{z \mid |z| \leq R\} \ \forall R$ så gælder

Så gælder $f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} (1 - z/a_n)$, g hel

og $f(z)$ har samme nulpkter med samme multiplicitet som $z^m \prod_{n=1}^{\infty} (1 - z/a_n)$

Bevis: Lad $R>0$ være givet og se på $\{z \mid |z| \leq R\}$

Så er alle faktorer $(1 - z/a_n) \neq 0$ for $n > N$ og $|z| \leq R$

Så har vi pr. antagelse $\prod_{\substack{n=1 \\ |a_n|>R}}^{\infty} (1 - z/a_n) \neq 0$ analytisk på $|z| \leq R$

$$h(z) = \prod_{n=1}^{\infty} (1 - z/a_n) = \prod_{|a_n| \leq R} (1 - z/a_n) \prod_{\substack{n=1 \\ a_n > R}}^{\infty} (1 - z/a_n)$$

$$\frac{h(z)}{\prod_{|a_n| \leq R} (1 - z/a_n)} = \underbrace{\prod_{|a_n|>R} (1 - z/a_n) \neq 0}_{\text{analytisk}} \quad \text{alle } z \in \{z \mid |z| \leq R\}$$

$$\frac{f(z)}{z^m \prod_{|a_n| \leq R} (1 - z/a_n)} = k(z) \neq 0 \quad \text{for } z \in \{z \mid |z| \leq R\}$$

vi har fået: $f(z) = z^m \prod_{|a_n| \leq R} (1 - z/a_n) k(z) \neq 0$ for $|z| \leq R$

$$= z^m \prod_{|a_n| \leq R} (1 - z/a_n) e^{g(z)} \quad g \text{ hel}, \ |z| \leq R$$

R_1, R_2, $R_1 < R_2$: $k(z) = e^{g(z)} = \dfrac{f(z)}{z^m \prod_{|a_n| \leq R} (1 - z/a_n)}$

Af entydighedssætn. følger $k_{R_1}(z) = k_{R_2}(z)$

$$\text{Res}\left(\frac{e^z}{z^2(z^2+5)}, 0\right) = \qquad , \; z^2+5 = (z+i\sqrt{5})(z-i\sqrt{5})$$

$$\text{Res}\left(\frac{1+z^2+z^2/2!+\ldots+z^4/4!+\ldots}{z^2(z^2+5)}, 0\right) =$$

$$\text{Res}\left(\frac{1}{z^2(z^2+5)} + \frac{1}{z(z^2+5)} + \ldots + \frac{1/2}{z^2+5} + \ldots\right) = \frac{1}{5}$$

$$f(z) = \frac{1}{z^2+5} \text{ analytisk i } 0$$

$$= \frac{1}{5} - \left(\frac{2z}{(z^2+5)^2}\right)_{z=0} z + f''(0)\frac{1}{2}z^2 + \ldots$$

$$= \frac{1}{5} + f''(0)z^2$$

$\pm i\sqrt{5}$ simple pole

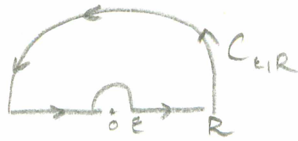
$$\text{Res}\left(\frac{e^z}{z^2(z^2+5)}, i\sqrt{5}\right) = \lim_{z\to i\sqrt{5}} (z-i\sqrt{5})\left(\frac{e^z}{z^2(z+i\sqrt{5})(z-i\sqrt{5})}\right)$$

$$= \frac{e^{i\sqrt{5}}}{(i\sqrt{5})^2(2i\sqrt{5})} = \frac{e^{i\sqrt{5}}}{-5\cdot 2i\sqrt{5}} = \frac{e^{i\sqrt{5}}}{-10i\sqrt{5}}$$

$$\text{Res}\left(\frac{e^z}{z^2(z^2+5)}, -i\sqrt{5}\right) = \frac{e^{-i\sqrt{5}}}{10i\sqrt{5}}$$

Uddybelse af foregående opg.

$$\int_0^\infty \frac{\log x}{1+x^2} = 0$$



$$\int_{C_{\varepsilon,R}} \frac{\log z}{1+z^2}\,dz = \int_{C_{\varepsilon,R}} \frac{\log z}{(z-i)(z+i)}\,dz = 2\pi i\ \mathrm{Res}\left(\frac{\log z}{(z-i)(z+i)}, i\right)$$

$$= 2\pi i\,\frac{\log_{-\pi/2} i}{2i} = \frac{2\pi i}{2i}(i\pi/2)$$

$$= i\,\frac{\pi^2}{2}$$

$$\log z = \log_{-\pi/2} z = \log|z| + i \arg_{-\pi/2} z \qquad -\pi/2 < \arg z \le 3\pi/2$$

$$z = x > 0 \quad \arg_{-\pi/2} z = 0 \quad , \quad z = -x < 0 \quad \arg_{-\pi/2} z = \pi$$

$$\int_{C_{\varepsilon,r}} \frac{\log z}{1+z^2}\,dz = I + II + III + IV$$

$$|II| = \left|\int_0^\pi \cdots\right| \le \varepsilon K \xrightarrow[\varepsilon\to 0]{} 0$$

$$\int_{-R}^{-\varepsilon} \frac{\log_{-\pi/2} z}{1+z^2}\,dz = \int_{-R}^{-\varepsilon} \frac{\log|z| + i\arg_{-\pi/2} z}{1+z^2}\,dz$$

$$\underset{z=-x}{=} \int_R^{-\varepsilon} \frac{\log(-x) + i\pi}{1+x^2}\,dx = \int_{-R}^{-\varepsilon} \frac{\log(-x)}{1+x^2}\,dx + i\pi \int_{-R}^{-\varepsilon} \frac{1}{1+x^2}\,dx$$

$$= \int_R^\varepsilon \frac{\log x}{1+x^2} - i\pi \int_R^\varepsilon \frac{1}{1+x^2}\,dx = \int_\varepsilon^R \frac{\log x}{1+x^2} + i\pi \int_\varepsilon^R \frac{1}{1+x^2}\,dx$$

$$\longrightarrow \int_0^\infty \frac{\log x}{1+x^2} + i\pi \int_0^\infty \frac{dx}{1+x^2} = \int_0^\infty \frac{\log x}{1+x^2}\,dx + i\frac{\pi^2}{2}$$

$$2\int_0^\infty \frac{\log x}{1+x^2}\,dx + i\pi^2/2 = i\pi^2/2$$

opgave

Lad f have en pol af orden 2 i z_0 og $h(z) = (z-z_0)^2 f(z)$

Så er $\operatorname{Res}(f, z_0) = h'(z_0)$

Bevis:

$$f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + g(z) \; ; \; g(z) \text{ analytisk nær } z_0$$

$$h(z) = (z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + g(z)(z-z_0)^2$$

$$h'(z) = a_{-1} + g'(z)(z-z_0)^2 + 2g(z)(z-z_0)$$

$$h'(z_0) = a_{-1} = \operatorname{Res}(f, z_0)$$

$$\operatorname{Res}\Big(\underbrace{\frac{e^z}{z^2(z^2+5)}}_{f(z)}, 0\Big)$$

$$h(z) = z^2 \frac{e^z}{z^2(z^2+5)} = \frac{e^z}{z^2+5} \quad , \quad h'(z) = \frac{e^z}{z^2+5} + e^z \frac{-2z}{(z^2+5)} = \frac{1}{5}$$

Husk uniform kontinuitet kan bruges til at vurdere

$$\left|\int_a^b (f(x) - f(x+i\varepsilon))\, dx\right|$$